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THE THEORY AND PRACTICE OF

ABSOLUTE MEASUREMENTS

IN

ELECTRICITY AND MAGNETISM

THE THEORY AND PRACTICE OF

ABSOLUTE MEASUREMENTS

IN

ELECTRICITY AND MAGNETISM

BY

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IN TWO VOLUMES

VOLUME I

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PREFACE.

THE present work is an extension of my small book which, published in 1884, has long been out of print. When the question of the issue of a second edition arose about three years ago, it was thought desirable to extend the plan of the former book so as to make the new work a fairly complete treatise on the absolute measurement of electric and magnetic quantities. It has not been my aim to produce a work dealing with mere manipulative processes or a collection of practical rules, but one in which should be welded together, in some degree at least, the practice of absolute measurements and the mathematical theory of electric and magnetic phenomena.

Thus it was no part of my plan to deal with the more recondite and abstract parts of electrical theory; but I trust the work now published may prove of some help to students who may wish to proceed to those subjects. I have, notably in Chapter I. of the present volume, included here and there for the sake of illustration particular theoretical cases which have no direct bearing on experimental processes. Most of

the purely theoretical work in electrostatics will however be of direct service in the sketch of magnetic theory to be given in Volume II. I have attempted throughout to arrange the work so as to avoid too sharp a distinction anywhere between what is theoretical and what is practical, such as might have been produced by giving all the theory in one part, and all the practical rules and processes in another. At the same time I have found it necessary for the preservation of logical order, and to prevent continual digressions on theory in the midst of descriptions of instruments and processes of measurement, to provide in each principal division of the subject a separate chapter containing the more general parts of the theory, leaving the more directly related theoretical questions to be treated, as I think they ought to be, where they arise.

In order to save space, the more mathematical parts of Volume I. have been printed in a somewhat smaller type than that adopted for the body of the work, and for the same reason the "solidus" notation has been generally used in formulæ occurring in the midst of ordinary matter. Here and there this latter practice has been inadvertently deviated from.

The scope of Volume I. will be seen in detail from the Table of Contents below; briefly, it consists of a sketch of the theory of electrostatics and flow of electricity, chapters on units, general physical measurements, electrometers, comparison of resistances, comparison of capacities, and measurement of specific inductive capacities, and concludes with tables of units,

resistances, and useful constants. The chapter on the comparison of resistances contains full details of the various methods of comparing high and low resistances, calibration of wires, &c.; the chapter on capacities discusses methods generally, and contains an account, as full as possible, of the principal determinations of specific inductive capacity made up to the present time. Determinations of dielectric strength, and other investigations regarding dielectrics, have on account of want of space been reluctantly omitted.

Volume II. will contain an account of magnetic theory, units, and measurements; electro-magnetic theory and absolute measurement of currents, potentials, and electric energy; the definitions and realisation of the ohm and other practical units; the relations of electro-magnetic and electrostatic units and the determination of v; practical applications of electricity, and specially related points of theory and measurement. In Volume II., on account of the great mass of matter included in the subjects here enumerated, the plan of smaller type will have to be adopted for descriptive and other details, as well as for mathematical theory.

For the use of blocks of woodcuts I am under obligations in the present volume to Sir WILLIAM THOMSON; Professor AYRTON, and the publishers of his *Practical Electricity*, Messrs. CASSELL; the late Professor BALFOUR STEWART and Mr. GEE; Dr. S. P. THOMPSON; the editor and publishers of the *Electrical Journal;* Messrs. ELLIOTT and Co.; and my publishers, Messrs. MACMILLAN and Co. I have received great

assistance from my brother, Professor T. GRAY, of the Rose Polytechnic Institute, Terre Haute, Ind., who allowed me to use his papers on "Electrical Testing," published in the *Electrical Journal* and elsewhere, and, besides reading nearly all the proofs, made many valuable suggestions. I have to acknowledge suggestions also from my colleagues, Professor G. B. MATHEWS, M.A., Fellow of St. John's College, Cambridge, and Mr. D. M. LEWIS, M.A., who have very kindly read proofs of various parts.

I have of course received continual help from the works of Sir W. THOMSON and CLERK-MAXWELL. Messrs. MASCART and JOUBERT'S recent valuable work, and Professor G. WIEDEMANN'S encyclopædic treatise with its wealth of references have been of much assistance. In all cases however in which it was possible recourse has been had to original papers, and in this connection I have to thank Dr. HOPKINSON, who favoured me with copies of papers on "Specific Inductive Capacity," and Professor AYRTON, who lent me copies of papers to which otherwise I should not easily have obtained access.

A few errata noticed in the preparation of the Table of Contents are given on page xxiv. I shall feel obliged if any reader who may find further errors will kindly communicate them either to my publishers or to myself.

A. GRAY.

UNIVERSITY COLLEGE OF NORTH WALES,
September, 1888.

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ERRATA.

Page 11, in marginal *for* "Creen" *read* "Green."

Page 20, line 6 from bottom, *for* "AC" *read* "CD."

Page 35, in marginal at bottom, *for* "in a soap-bubble" *read* "on a soap-bubble."

Page 39, line 10 from top *insert* "at" *before* "any."

Page 41, line 3 from bottom, *for* "in C_s" *read* "on C_s."

Page 60, last line, *for* "IV." *read* "VII."

Page 80, line 12 from bottom, *for* "$-q$" *read* "$-q.a/f$."

Page 104, in first marginal, *for* "Measure of Heat" *read* "Measure of Flux of Heat."

Page 144, line 12 from top, *for* "Chapter VIII." *read* "Volume II."

Page 165, equation (31), *for* $\frac{r_1 + r_2}{8k\, r_1\, r}$ *read* $\frac{r_1 + r_2}{8k\, r_1\, r_2}$.

Page 203, line 12 from bottom, *for* "(K)" *read* "(C)."

Page 204, foot-note, *for* "Chapter V." *read* "Chapter VI."

Page 225, transfer marginal to first paragraph of next page.

Page 234, in first marginal, *for* "Form of Vibration" *read* "Form of Vibrator."

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ABSOLUTE MEASUREMENTS

IN

ELECTRICITY AND MAGNETISM.

CHAPTER I.

ELECTROSTATIC THEORY.

SECTION I.

ELECTRIC ATTRACTION AND REPULSION. ELECTRIC POTENTIAL.

WE suppose the reader to be familiar with the elementary phenomena of electricity, and that he has acquired a clear understanding of what is meant by *quantity of electricity*, and of the nature of the evidence for the truth of the fundamental law of attraction and repulsion. Some account of the theoretical bearing of the experimental results of Coulomb, Cavendish and Faraday, in statical electricity is given below, but recourse must be had, if possible, to the original memoirs,[1] or failing these to some good treatise, for the details of the investigations.

[1] Faraday's *Experimental Researches*, Cavendish's *Electrical Researches*, and Coulomb's *Memoirs* (in French), have been collected and reprinted.

Law of Electric Attraction and Repulsion.

The law of electric attraction and repulsion may be stated as follows: The force between two quantities of electricity (supposed each concentrated at a point) is directly proportional to the product of the charges, and inversely proportional to the second power of the distance between them. If the charges are unlike the mutual force is an attraction, if of the same kind a repulsion. In symbols, if q, q' be the charges, r the distance between the points at which they are concentrated, and F the mutual force between them, we have

$$F = k\,\frac{qq'}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where k is a multiplier which does not vary with the other quantities, and depends on the units adopted, and on the medium surrounding the charges. We shall take the force F as positive when repulsive, and therefore k as a positive quantity. We shall suppose, unless it is otherwise stated, that the phenomena take place in a perfect vacuum.[1]

Coulomb's Experiments.

This law was experimentally established with approximate accuracy by Coulomb by means of his torsion balance, in which he measured against the elastic reaction of a twisted silver wire the mutual attraction or repulsion between small charged conductors placed at a measured distance apart. He found that when the charges remained the same and the distance was doubled the force between the conductors was reduced approximately to $\frac{1}{4}$ of its former amount; and in general, that the force for the same charges varied inversely as the square of the distance.

[1] That is a space containing *ether* only.

Having measured in his balance the force between two small spherical conductors, Coulomb withdrew one of them and brought it into contact with another conducting sphere of the same diameter, then replaced it, and brought the balls to the same relative position as before. The force between the two spheres was now one-half of its former amount. Now by the contact the two balls must by symmetry have received equal charges, and Coulomb with propriety assumed that the charge on the ball was one-half of the initial charge. He thus obtained the result that the mutual attraction or repulsion of two conductors of linear dimensions small in comparison with the distance between them is directly proportional to the product of their charges when the distance is maintained the same. This connection between force and amount of charge lies at the foundation of our system of measuring quantities of electricity.

Coulomb's Experiments: Quantity of Electricity.

In the electrostatic system of units, which we shall find it convenient to use in the present chapter, k in equation (1) is taken equal to 1, so that when q, q', r are each 1, F is also 1. Now *unit force* is defined according to what is called the *absolute* system of measurement of forces founded on Newton's Second Law of Motion, as *that force which acting for unit of time on unit of mass will give to that mass unit velocity.* Hence unit quantity of electricity is that quantity which concentrated at a point at unit distance from an equal quantity of the same kind is repelled with unit force. Unit quantity of electricity, therefore, depends on the three fundamental

Kinetic or Absolute Unit of Force.

Unit Quantity of Electricity.

C.G.S. system of Units.

units of length, mass, and time. According to the recommendations of the British Association Committee on Electrical Standards, and the resolutions of the Paris Congress of Electricians held in 1882, it has been resolved to adopt generally the three units already in very extended use for the expression of dynamical, electric and magnetic quantities; namely, the Centimetre as the unit of length, the Gramme as the unit of mass, and the Second as the unit of time, and these units are designated by the letters C.G.S. With these units therefore unit force is that force, which, acting for one second on a gramme of matter, generates a velocity of one centimetre per second. This unit of force has been called a *Dyne.* Unit quantity of electricity in the C.G.S. system of units is, accordingly, that quantity which placed at a distance of 1 centimetre from an equal quantity is repelled with a force of 1 dyne.

The following example, which is easily realised, is instructive as an illustration of this definition and of the idea of quantity of electricity.

Two small equal pith balls are hung by very fine silk fibres from a fixed point so as to form two similar pith ball pendulums hung side by side. The balls are charged simultaneously by contact with the same conductor and the pendulums then diverge. It is required to find the electric charge in each ball, and to work out numerically for the case of each ball $\frac{1}{40}$ gramme in mass, length of fibre 80 centimetres, and distance of the centres of the balls apart 10 centimetres.

Let l be the length of each fibre, $2d$ the distance

between the centres of the balls, m the mass, and q the charge of each ball.

Unit Quantity of Electricity.

Neglecting the inductive effect of the balls on one another, and the weight and any electrification of the

fibres, we have for the mutual force between the balls $q^2/4d^2$, and by the parallelogram of forces

$$\frac{q^2/4d^2}{mg} = \frac{d}{\sqrt{l^2 - d^2}} = \frac{d}{l\sqrt{1 - d^2/l^2}}$$

or

$$q^2 = \frac{4d^3\,mg}{l\sqrt{1 - d^2/l^2}}$$

In the example given we have $m = \frac{1}{40}$, $l = 80$, $d = 5$; and taking g as 981 we get, neglecting d^2/l^2

$$q^2 = \frac{12{\cdot}5 \times 981}{80}$$

or

$$q = \pm\, 12{\cdot}38.$$

The charge on each ball is therefore approximately 12·38 C.G.S. units of positive or negative electricity.

Electric Field.

The *Electric Field* of any distribution of electricity is the whole surrounding space through which the action of the electrified system extends, and the electric force at any point is the force of attraction or repulsion which a unit of positive electricity would experience if placed at the point without disturbing the electric distribution of the system.

Intensity of Electric Field.

The *Intensity* of an electric field at any point is measured by the electric force at that point. We may imagine the electric field to extend to an infinite distance from any part of the electrified system, and for infinitely distant points the electric force is of course zero.

Let the electric system consist of a quantity q of electricity at a point whose coordinates are a_1, b_1, c_1; a quantity q_2 at a point a_2, b_2, c_2, &c., ; and let the coordinates at any point P in the field be x, y, z. The distance r, of x, y, z, from any point a, b, c, is given by the equation $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$. The electric force at P has then for its three components

$$X = \frac{q_1 (x - a_1)}{r_1^{\,3}} + \frac{q_2 (x - a_2)}{r_2^{\,3}} + \text{\&c.}$$

$$\left.\begin{aligned} \text{or} \qquad X &= \Sigma \frac{q (x - a)}{r^3} = \Sigma \frac{q}{r^2} \frac{dr}{dx}. \\ \text{Similarly} \quad Y &= \Sigma \frac{q (y - b)}{r^3} = \Sigma \frac{q}{r^2} \frac{dr}{dy} \\ Z &= \Sigma \frac{q (z - c)}{r^3} = \Sigma \frac{q}{r^2} \frac{dr}{dz}. \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

where Σ denotes summation for all values $a_1, b_1, c_1, a_2, b_2, c_2$, &c., of a, b, c.

The resultant electric force F is given by the equation

$$F^2 = X^2 + Y^2 + Z^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

A line drawn in the electric field, the tangent to which at any point is the direction of the resultant force at that point, is called a *Line of Force*. Definition of Line of Force.

If dx, dy, dz, be the projections on the axes of x, y, z of an element ds of a line of force, and X, Y, Z be the components of the force there parallel to the axes, we must have the relation Differential Equation of a Line of Force.

$$\frac{X}{dx} = \frac{Y}{dy} = \frac{Z}{dz} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3\ bis)$$

which is the differential equation of a line of force. We shall show below in treating of lines of magnetic force how this equation may be integrated for the case of a force system symmetrical round an axis.

The *Electric Surface Density* at any point of a surface is the limit towards which the ratio of the quantity of electricity on an element of the surface including the point to the area of the element approaches as the element is taken smaller and smaller. Surface Density of Electric Distribution.

Let σ be the surface density at a point a, b, c of an electrified surface and ds the area of an element including the point, then instead of (2) we have

$$\left.\begin{aligned} X &= \iint \frac{(x-a)\,\sigma ds}{r^3} = \iint \frac{\sigma ds}{r^2}\frac{dr}{dx} \\ Y &= \iint \frac{(y-b)\,\sigma ds}{r^3} = \iint \frac{\sigma ds}{r^2}\frac{dr}{dx} \\ Z &= \iint \frac{(z-c)\,\sigma ds}{r^3} = \iint \frac{\sigma ds}{r^2}\frac{dr}{dz} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $\iint$ denotes integration over the surface.

The *Electric Volume Density* at any point in space is the limit towards which the ratio of the quantity of electricity contained within an element of space Volume Density of Electricity

Volume Density of Electricity

including the point to the volume of the element approaches as the element is taken smaller and smaller.

Let ρ be the volume density at a point a, b, c, the quantity of electricity within an element of volume $da\,db\,dc$ is $\rho\,da\,db\,dc$. Hence we have

$$X = \iiint \frac{(x-a)\,\rho\,da\,db\,dc}{r^3} = \iiint \frac{\rho}{r^2}\frac{dr}{dx}\,da\,db\,dc \quad . \quad (5)$$

with similar formulas for Y and Z, where $\iiint$ denotes integration throughout the space or spaces occupied by the electric distribution. In every case the intensity F of the electric field at any point is given by the equation

$$F^2 = X^2 + Y^2 + Z^2.$$

Electric Potential.

The *Potential* at a point in an electric field is the work done by or against electric forces in carrying a unit of positive electricity from the point in question to an infinite distance, the electric distribution being supposed to remain unchanged. Hence if a quantity q of positive electricity be concentrated at a point O, and P be a point at distance r from O, the potential at P due to q is $\frac{q}{r}$.

For the force of repulsion on a unit of positive electricity at a distance x from O is $\frac{q}{x^2}$; and the work done by this force in increasing the distance by a small length dx is $\frac{q}{x^2}\,dx$. Hence if V be the potential at P we have

$$V = \int_r^\infty \frac{q}{x^2}\,dx = \frac{q}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Difference of Potentials.

The *Difference of Potentials* between P and another point P' at a distance r' from O, or the work done in

carrying a unit of positive electricity from P' to P, is therefore $\frac{q}{r} - \frac{q}{r'}$. It is important to remark that this value is independent of the path pursued between P' and P. It depends only on the distances of the points from O. Difference of Potentials.

Further, if we have a number of quantities q_1, q_2, q_3, &c. of electricity at distances r_1, r_2, r_3, &c. from P, the potential at P is $\frac{q_1}{r_1} + \frac{q_2}{r_2} +$ &c., or, as it is usually written, $\Sigma \frac{q}{r}$, where Σ denotes summation of a series of terms of the form $\frac{q}{r}$. Hence the difference of potentials between P and P' is in this case $\Sigma \frac{q}{r} - \Sigma \frac{q'}{r'}$. The value of $\Sigma \frac{q}{r} - \Sigma \frac{q'}{r'}$, depends only on the *positions* of the quantities $q_1\, q_2$ &c.

If the distribution of electricity be continuous over any surface, or throughout any space, the summation becomes integration over the surface, or throughout the space. In the former case we have for V the equation

$$V = \iint \frac{\sigma ds}{r}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in the latter

$$V = \iiint \frac{\rho\, da\, db\, dc}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

and if the distribution be of both kinds we have

$$V = \iint \frac{\sigma ds}{r} + \iiint \frac{\rho\, da\, db\, dc}{r} \quad . \quad . \quad . \quad . \quad (9)$$

where $\iint$, $\iiint$ respectively denote as before integration over

Difference of Potentials.

the surface and throughout the space occupied by the distribution.

Since $r^2 = (x - a)^2 + (y - b)^2 + (z - c)^2$ we see from equations (2) that

$$X = -\frac{dV}{dx},\ Y = -\frac{dV}{dy},\ Z = -\frac{dV}{dz} \quad . \quad . \quad . \quad (10)$$

that is, the force-components parallel to the axes at any point x, y, z are equal numerically to the rates of variation of the potential in the respective directions.

Further if we calculate the force variations in their respective directions by differentiation in (2), and add, we obtain the remarkable relation

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 0. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Laplace's Equation.

This result is called Laplace's Equation. It only holds when there is no electricity at the point x, y, z. We shall prove a more general equation given by Poisson; but first we shall establish the following proposition, from which can be deduced many interesting results as to attractions or repulsions in different cases :—

Let N denote the outward normal component of the resultant electrical force at a point P situated on a small element (of area ds) of a closed surface in an electric field: the sum of all the products Nds obtained by dividing the surface into small elements and multiplying the area of each by the value of N at the element, or the value of $\iint Nds$ taken over the surface, is numerically equal to the whole quantity of electricity contained within the surface multiplied by 4π.

Surface Integral of Electric Induction.

The product Nds has been called by Maxwell the *Electric Induction over the element ds,* and $\iint Nds$ the *electric induction over the surface.* The algebraic sum of the quantities of electricity within the surface is

therefore equal to the electric induction over the surface divided by 4π.

Surface Integral of Electric Induction: a Theorem of Green.

Let S (fig. 2) be the surface and first consider the normal force at A due to a quantity q of electricity concentrated at O. Let a cone of small vertical angle be described with its apex at O by drawing lines all passing through O, and through the periphery of a small element ds of the surface at A. Let ϵ be the angle which OA makes with the normal drawn outwards at A and let r be the distance OA. We have, if R be the resultant force at A due to q, $R = \frac{q}{r^2}$; and if N

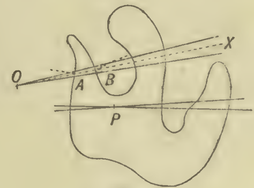


Fig.2

be the normal force at ds

$$Nds = R\cos\epsilon\, ds = \frac{q}{r^2}\cos\epsilon\, ds.$$

Now if $d\omega$ be the area intercepted by the cone on a spherical surface of unit radius described from O as centre, we have

$$ds\cos\epsilon = -r^2 d\omega,$$

and therefore

$$Nds = -qd\omega.$$

Now considering the element ds' of the surface intercepted at B by the cone in emerging we have

$$N'ds' = R\cos\epsilon'\, ds' = qd\omega.$$

Hence for these two elements

$$Nds + N'ds' = 0.$$

If as shown in the figure the cone enter and emerge more than once, this equation must hold for each pair of elements corresponding to an entrance and emergence, and hence for all the elements intercepted by the cone we have $\Sigma Nds = 0$. It is

Surface Integral of Electric Induction.

evident that by drawing cones in this way we could divide the whole surface up into pairs of elements, and therefore so far as the quantity q of electricity external to the surface is concerned

$$\iint Nds = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Since we can draw cones from every point, this must hold for any external distribution.

Now considering a quantity q of electricity at a point P within the surface, we can show as before by drawing cones that for any element

$$Nds = \begin{array}{l} +qd\omega, \text{ for emergence.} \\ -qd\omega, \text{ for entrance.} \end{array}$$

But as the cones all originate within the surface, and therefore emerge as shown once oftener than they enter, $qd\omega$ is the value of the part of $\iint Nds$ which corresponds to each case. Therefore

$$\iint Nds = q \iint d\omega.$$

But $\iint d\omega$ is simply the area of the spherical surface of unit radius, that is 4π; therefore

$$\iint Nds = 4\pi q,$$

and since we can go through the same process for every part of the internal distribution

$$\iint Nds = 4\pi M \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where M is the whole quantity of electricity within the surface. From the result for the external electrification we see that if N be the normal component at any element ds due to the electrification both internal and external, we have the proposition stated above.

Maximum or Minimum of Potential in Free Space Impossible.

From this proposition it follows that there cannot be a point of maximum or a point of minimum potential in space void of electricity, for if there were the potential would in one case diminish and in the other increase in every direction from the point, and the electric induction over a closed surface including the point would not be zero.

Let the closed surface be a small rectangular parallelepiped

of sides dx, dy, dz, with its centre at O, and let X_1 be the normal component at the side $dydz$ to the left of the origin, and X_2 that at the opposite side. The part of the surface integral given by the sides is $(X_2 - X_1)\, dy\, dz$.

Characteristic Equation of the Potential.

Adding to this the parts due to the other two pairs of sides we find

$$\frac{X_2 - X_1}{dx} + \frac{Y_2 - Y_1}{dy} + \frac{Z_2 - Z_1}{dz} = 4\pi\rho \quad . \quad . \quad (14)$$

where ρ is the average volume density of the distribution within the surface.

It is easy to show from equations (2) above, by changing to polar coordinates, that the values of X, Y, Z vary continuously whether or not the points be in any part of the electrification, provided only the volume density of the distribution does not become infinite. We may write therefore, putting X, Y, Z, for the component forces at O,

$$X_1 = X - \tfrac{1}{2}\frac{dX}{dx}dx, \quad X_2 = X + \tfrac{1}{2}\frac{dX}{dx}dx,$$

and similar formulas for Y_1, Y_2, Z_1, Z_2. Equation (14) becomes therefore

Poisson's Equation.

$$\frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz} = 4\pi\rho \quad . \quad . \quad . \quad . \quad . \quad (15)$$

or, writing $-dV/dx$, $-dV/dy$, $-dV/dz$, instead of X, Y, Z,

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + 4\pi\rho = 0 \quad . \quad . \quad . \quad . \quad (16)$$

This equation is due to Poisson and is called sometimes the characteristic equation of the potential. When $\rho = 0$, we obtain Laplace's equation, which is therefore a particular case of (16).

By a change of sign in all *forces* in the equations the theorems proved above and others which follow, can be made applicable to gravitational attraction.[1] For example let it be required to find the attraction of a uniform sphere or spherical shell of attracting matter on an external particle P of unit mass at a distance r from the centre. By symmetry, the attraction at every point of the spherical surface concentric with the sphere or shell and passing through P is the same as at P and normal

Attraction (or repulsion) of a Sphere or Shell on an External Particle.

[1] In gravitational attraction the potential is defined as $\Sigma\,(q/r)$, and the force parallel to any direction x taken positive in the direction of x increasing, is then dV/dx.

Attraction (or repulsion) of a Sphere or Shell on an External Particle.

to the surface. Hence by the theorem proved above, if F be the force

$$-F \times 4\pi r^2 = 4\pi M,$$

or

$$F = -M/r^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

that is the attraction is the same as if the whole mass of the sphere or shell were collected at the centre.

By equation (6) $V = \frac{M}{r}$, or the potential is also the same as if the whole mass were collected at the centre.

Attraction of Spherical Shell on Internal Particle.

Again, let the particle P be within a hollow shell bounded by two concentric spherical surfaces; the attraction at every point of a spherical surface passing through P and concentric with the shell must be the same and normal to the surface. Calling it F, and the area of the surface S, we get

$$F \times S = 0, \quad \text{or} \quad F = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

that is the force is zero within such a shell.

Potential Constant within uniform Spherical Shell.

Since the force $\frac{dV}{dr}$ is zero the potential is constant within the shell, and is therefore everywhere what it is at the centre. The potential of a uniform spherical shell of radius x, thickness dx, and density ρ, is $4\pi\rho x^2 dx/x = 4\pi\rho x dx$. Hence the potential in the interior of a uniform spherical shell of internal radius a and external radius r is

$$4\pi\rho \int_a^r x dx = 2\pi\rho\,(r^2 - a^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Potential and Force at any point within a uniform Sphere.

The part of the potential at a point in a homogeneous sphere at a distance a from the centre due to the external shell is $2\pi\rho\,(r^2 - a^2)$, and the part due to the sphere of radius a on the surface of which the point is situated is $\frac{4}{3}\pi\rho a^2$. Hence

$$V = 2\pi\rho\,(r^2 - \tfrac{1}{3}a^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Imagine a concentric spherical surface described through the point. The force, F, at the surface is everywhere normal to it, and if a be the radius we have

$$-F \times 4\pi a^2 = 4\pi \frac{4\pi\rho}{3} a^3,$$

or

$$F = -\frac{4\pi}{3}\rho a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

that is, the force varies as the distance from the centre. This could of course be at once obtained by finding $\frac{dV}{da}$, in equation (20) above.

As another example consider an electrified circular disc of uniform electric density σ and radius r acting on a unit of positive electricity placed on the axis of the disc at a distance h from its plane. The potential of a concentric ring of radius x and breadth dx is $2\pi\sigma x dx/\sqrt{h^2+x^2}$. For the whole disc therefore

Force at point on axis of Electrified Disc.

$$V = \int_o^r 2\pi\sigma x dx/\sqrt{h^2+x^2}$$

$$= 2\pi\sigma\left(\sqrt{h^2+r^2} - h\right) \quad . \quad . \quad . \quad . \quad . \quad (22)$$

For the repulsion on the particle we have the value

$$-dV/dh = 2\pi\sigma\left(1 - h/\sqrt{h^2+r^2}\right) \quad . \quad . \quad . \quad (23)$$

The following proposition (due to Gauss) is interesting and important: The average potential over a spherical surface due to electricity entirely without the sphere is equal to the potential at the centre.

Average Potential over Sphere due to External Charge.

Suppose the electric distribution to be a quantity q situated at a point O. The potential at a point P on the spherical surface distant r from O is $\frac{q}{r}$. The average potential over the surface is, if R be its radius, $\frac{1}{4\pi R^2}\iint \frac{q}{r}\,ds$. But $\iint \frac{q}{r}\,ds$ is the potential at O due to a uniform distribution of surface density q over the spherical surface and is therefore (p. 14) $\frac{4\pi R^2 q}{D}$, if D be the distance of O from the centre of the sphere. The average potential is therefore $\frac{q}{D}$, that is, the potential at the centre of the sphere due to q at O. The same result can be obtained for every part of the distribution, and the proposition is established.

From this theorem follows obviously the result already proved that there is no place of maximum or of minimum potential in space void of electricity.

Average Potential over Sphere due to External Charge.

Further it follows if throughout any space S in which there is no electricity the potential be constant, the potential must have the same value throughout all space that can be reached from S by any path which does not pass through the electrified system. For if throughout any space S', adjoining S, the potential be greater or less than in S, it must be possible to describe a sphere so small that its centre and a portion of its surface may be in S, and the rest of its surface in S'. The mean potential over the surface would then be either greater or less than the potential at the centre, which is impossible. Hence S' cannot have at any point a greater or less potential than that of S.

Equipotential Surfaces.

A surface every point of which is at the same potential is called an *Equipotential Surface*, or sometimes a *Level Surface*. Such a surface can evidently be drawn for every point of the electric field.

Surface of Zero Potential.

Any equipotential surface may be taken as the surface of zero potential, and the potential at any point is then simply the difference of potentials between the point and that surface. The potential of the earth is generally taken as zero potential.

Since no work is done in carrying a unit of positive electricity from one point of an equipotential surface to another, lines of force meet such surfaces at right angles; in other words, the direction of the resultant force at any point of such a surface is normal to the surface.

The potential within a closed equipotential surface which contains no electricity is constant. For if not there must be an adjacent equipotential surface within,

at a higher or lower potential, and the electric induction over every element of that surface must have the same sign. But the integral over the whole surface is zero, hence it is zero over every element, that is, the electric force at every element is zero. There is therefore no difference of potential between the two surfaces; and by applying the theorem of Gauss (p. 15 above), or by considering successive internal surfaces we can prove that the whole internal space is at the same potential.

Surface of Zero Potential.

It is necessary for electrical equilibrium that the electric force at every point of the substance of any conductor, whether containing within it electrified bodies or not, be zero, that is, that the potential at every point be the same. Any surface therefore described within the substance of a conductor is an equipotential surface, and, by the proposition just proved, the potential must have the same value at every point within a hollow conductor containing no electrified bodies, as at the conductor itself.

Potential Constant within Hollow Conductor.

Since the electric force is everywhere zero there is no free electricity at any point in the substance of the conductor, for no lines of force can enter or leave any portion of space there situated. Hence the charge of a conductor, can be found only on the external surface, and if a conductor in the interior be brought into contact with the internal surface it will, if electrified, give up its whole charge to the external conductor, and if unelectrified receive no charge.

Charge of Conductor only on Surface.

These conclusions, which have been deduced from the law of electric attraction and repulsion stated above (p. 1), are in accordance with the results of experiment.

Experimental Proofs.

Experimental Proofs.

A charged conductor introduced within a closed conductor and brought into contact with it as in Faraday's ice-pail experiment (p. 22), and in Biot's well-known experiment in which two insulated hemispherical conductors are made to enclose and touch a highly charged metallic sphere, is found to be completely discharged. The distribution otherwise than on the external surface is therefore not one of equilibrium.

Cavendish's Experiment.

The same result was found so long ago as 1773 by the Hon. Henry Cavendish, who used the other form of experiment. A conducting sphere was insulated within a concentric spherical shell made of two hemispherical conductors mounted on insulating supports so as to be easily removable. The shell after having been highly charged was brought into contact for an instant with the internal sphere. The hemispheres were then removed and discharged, and the electrical state of the sphere tested by means of a delicate electroscope. No trace of a charge could be detected.[1]

Experimental Proof of Minor Premiss of

The tests of such a charge now afforded by Sir William Thomson's Quadrant Electrometer (Chap. II. below) exceed enormously in delicacy any which could previously be applied, and careful repetitions of the experiment made with the help of that instrument have shown no inaccuracy in Cavendish's result.[2]

Cavendish's Hypothetical Syllogism.

This result is of the utmost importance, for, assuming it as a fact experimentally proved, we can reason back from it to the law of the inverse square for electric

[1] Cavendish's *Electrical Researches*, or Maxwell, *Elect. and Mag.* vol. i. p. 77 (sec. ed.).

[2] Maxwell, *Elect. and Mag.* (sec. ed. vol. i. p. 78).

attraction and repulsion, which Coulomb arrived at by the immensely more uncertain method of directly measuring, in the torsion balance, the attraction and repulsion between electrically charged conductors. Cavendish's experiment is therefore properly regarded as the crucial test of the law of the inverse square.

Cavendish's Hypothetical Syllogism stated.

We shall now prove briefly that if, as shown by Cavendish's experiment, the electric distribution on a charged spherical conductor is entirely on the external surface, or, which is the same thing, if there be zero force everywhere within the conductor, the law of the inverse square is true.[1] If the sphere be at a very great distance from all other conductors the distribution on it must from symmetry be uniform, and we have seen (p. 14 above) that the law of the inverse square satisfies the condition of zero force in the interior. It remains to show that this is the only law which is consistent with a distribution entirely on the surface.

Mathematical Proof of Major Premiss.

Let $F(r)$ denote the electric force at a distance r from a unit of positive electricity concentrated at a point; then $F(r)$ fulfils the condition stated. In the case of the law of the inverse square $r^2 F(r)$ is a constant, for any other law it is not. If $r^2 F(r)$ is not constant, it must for any given value of r either increase or diminish as r is increased. Let r_1 and r_2 be any two values of r such that $F(r)$ continuously increases as r increases from r_1 to r_2. Let $ABCD$ (Fig. 3) be a great circle of a spherical conductor of diameter $r_1 + r_2$ and surface

[1] The proof here given is due to M. Bertrand, *Journ. de Phys. t.* II. p. 41 (1873). For other proofs see Laplace, *Mécanique Celeste,* i. 2; Cavendish, *Elect. Res.*; Maxwell, *loc. cit.*

Proof of Major Premiss of Cavendish's Syllogism.

density σ, P a point such that $CP = PD$, and AB the diameter drawn through P. Draw a double cone of small vertical angle $d\omega$ at P and denote by θ the angle which the axis of the cone makes with the radius at its intersections with the sphere. The quantities of electricity on the opposite elements of the sphere are, if r'_1, r'_2 are their distances from P, $\sigma\ r'^2_1\ d\omega/\cos\ \theta$, $\sigma\ r'^2_2\ d\omega/\cos\ \theta$ respectively, and the components of their attractions along the diameter

$$\sigma\, d\omega\, r_1^2\, F(r_1) \cos\phi/\cos\theta, \quad \sigma\, d\omega\, r_2^2\, F(r_2) \cos\phi/\cos\theta,$$

where ϕ is the angle between the axis of the cone and AB, as shown. These attractions are in opposite

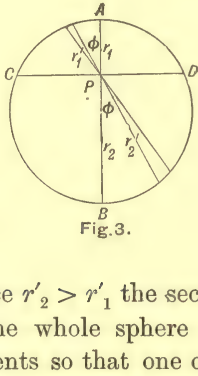


Fig. 3.

directions, and since $r'_2 > r'_1$ the second is greater than the first. Now the whole sphere can be divided up into pairs of elements so that one of each (of distance r'_1), lies above AC, and one (of distance r'_2), beneath: hence the attraction of P towards A must be greater than that towards B, that is, the condition that there is no force within the conductor is not fulfilled. Hence the hypothesis that $r^2\, F(r)$ increases as r increases from r_1 to r_2 cannot be true; and in the same way it can be

shown that $r^2\,F(r)$ does not diminish as r increases between any limits whatever. It follows that $r^2\,F(r)$ is a constant if the electric distribution is wholly on the surface.

Faraday's Electrified Cube.

Cavendish's result was incidentally confirmed and strikingly illustrated by Faraday in his researches on the "Absolute Charge of Matter."[1] Having constructed a cubical framework of wood (ten feet in length of edge), covered with a network of copper wire and bands of tinfoil, he insulated it and placed within it a very delicate gold leaf electroscope. He then powerfully electrified the cube, and found that the electroscope showed no effect. He even went into the cube and lived in it, and tried without effect all the tests of electrification he could apply, although all the time long sparks and brush discharges were passing from its outer surface.

Induction within Closed Conductor:

We can now prove simply the important result that if a closed conductor contain any electrified system having a total charge Q, a total charge of electricity $-Q$ is induced on the inner surface of the conductor. For suppose a surface S described within the substance of the conductor C intermediate between its inner and outer surfaces, then since there is no force within the substance of C the electric induction over S must be zero. Hence the total quantity of electricity within S must be zero, that is, the proposition stated above must be true. We shall see below that the density of the induced distribution at any point depends on the distribution of the internal inducing system.

Internal Induced Charge.

[1] *Experimental Researches*, vol. i. p. 366.

External Induced Charge.

If C be insulated without charge it follows that a charge equal to $+ Q$ is found on the external surface. We shall see below (p. 26) that the distribution of this charge is independent of that of the internal charges, and is the same as that of a free charge of the same amount given to the conductor.

Faraday's Ice-pail Experiment.

These results give the explanation of Faraday's ice-pail experiment alluded to above. A pewter ice-pail was supported by silk threads and connected to a delicate gold leaf electroscope. A charged conducting ball was then lowered into the ice-pail, and as it descended the gold leaves gradually separated until the ball had been placed well down in the pail, when the divergence remained nearly constant as the ball was lowered further, brought into contact with the pail, and withdrawn. The ball was then found to have been totally discharged.

The same experiment can be repeated with exceedingly great delicacy by means of the quadrant electrometer, and if the pail is well insulated and closed by a conducting cover to which the ball is hung by an insulating support kept free from electrification, the result described is easily obtained. The insulating support if electrified may be discharged by being passed through a flame, or by being placed in a current of hot air. With a quadrant electrometer in its ordinary grade of sensibility it is necessary to charge the ball very slightly. This can be done without sensibly electrifying the insulating support by giving a small spark by means of an electrophorus to a second ball held by an insulating handle, then touching that ball with a third also insulated, dis-

charging the latter and again bringing it into contact with the second, and so on until the charge of the second ball is so much diminished that the experimental ball brought into contact takes a charge just sufficient to give a convenient deflection of the electrometer needle. The constancy of the electrometer deflection after the contact shows that the potential of the conductor is not altered, and it follows as proved below (p. 26) that the distribution on the exterior surface of the conductor is not changed. The total discharge of the ball proves that its charge was equal and opposite to the induced charge on the interior surface of the conductor.

Faraday's Ice-pail Experiment.

This experiment is the basis of many useful electrical instruments, notably induction machines such as Thomson's Replenisher, and the machines of Holtz, Voss, and others, which multiply electric charges; and it gives more clearly than any other the idea of quantity of electricity. For example it gives us a means of charging a conductor with any number of times a given electric charge. For let the charge be given on the ball of the ice-pail in the experiment above. The ice-pail is insulated within the conductor to be charged, and the ball placed within the ice-pail but without touching it. The ice-pail and the exterior conductor are then brought into contact for an instant, and the ball and ice-pail withdrawn. Since the ice-pail and conductor, when in position after the contact, are at one potential, there is no electrification between the inner surface of the former and the outer surface of the latter. Hence there is then a charge on the outer surface

Method of Multiplying Electric Charges.

Method of Multiplying Electric Charges.

of the conductor equal to, and of the same sign as, that on the ball. If the ice-pail, which is left with a charge equal and opposite to that on the ball, be discharged, and the process above described repeated, another charge equal to that on the ball will be given to the conductor, and so on until the required multiple of the charge has been given.

By slightly modifying this process a second conductor may be charged with an equal quantity of the opposite electricity. For it is only necessary to discharge the ice-pail each time by placing it within and then bringing into contact with the second conductor. If the opposite kind of electricity only is required it is sufficient to connect the ice-pail to earth each time the ball is placed within it.

Relation of Distributions Internal and External to a Closed Conductor

The results of Faraday's ice-pail experiments are in complete accordance with theory, and are direct consequences of the following general proposition regarding closed conductors. Whatever be the electrification of the closed conductor, the external electric distribution and the potential at every external point is the same for a given total quantity of electricity within the conductor without regard to the manner of its distribution: and the electric distribution on bodies within the closed conductor is independent of the electrification both of the external surface of the closed conductor itself and of external bodies. In the proof of this proposition, and frequently in what follows, we shall use the principle of *superposition of electric distributions*, which ought therefore to be first explicitly stated.

According to the theory given above (p. 9), the

potential at any point due to a charge of electricity at another point is directly proportional to the charge, and the potential at any one point produced by any distribution of electricity is the sum of the potentials due to the separate parts of which the distribution consists, or of the separate distributions into which it may be supposed divided.

Principle of Superposition stated.

An electric system in fact may be built up in any way whatever of separate parts or made up of separate distributions superimposed; each part of the system or each separate distribution produces its potential at any point as if the remainder did not exist; the final distribution is the sum of the separate distributions, and the final potential at any point the sum of their separate potentials. This conclusion is capable of direct verification by experiment in certain cases, and further the results deduced from it are found to agree with observed phenomena.

It is proved below (p. 69) that electricity can be distributed in one and only one way on a given system of conductors so as to produce a given possible system of potentials. Hence if by the superposition of different states of the same conductors we can produce the required potentials we know that we have obtained the only solution of the problem.

Relation of Distributions Internal and External to a Closed Conductor.

(1) Suppose no external electrified bodies to exist, and the closed conductor to be at zero potential, then, since the potential is zero also at a very great distance, there can be no potential greater or less than zero in intermediate space, otherwise there would be a maximum or minimum of potential in space unoccupied

Relation of Distributions Internal and External to a Closed Conductor

by electricity. Hence the distribution of electricity on the interior surface of the conductor is such as to produce through external space a potential exactly equal and opposite to that which the charged body produces.

(2) Suppose an external electrified system to exist, but no electricity to be within the conductor, the potential within the conductor (which may now be supposed insulated and charged to any degree) will be constant and of the same value as that of the conductor. The induced electrification of the conductor, as may be seen by supposing the potential zero, is such as to produce a potential within the conductor equal and opposite to that produced by the external electrified bodies; and we can superimpose on this the independent electrification, if any, which is effective in producing the actual potential of the conductor.

We have therefore in (1) and (2) three electrifications which are separately in equilibrium and may be supposed superimposed, and since, as we shall see below, there can be only one distribution corresponding to given potentials or charges of the system, the superimposed distributions are in equilibrium and form the actual distribution. Hence the proposition stated above is true; and we have also the very important result that if the conductor be connected with the earth it forms a perfect electrical screen between the internal and external systems. To protect an electric system from all external influences it is therefore only necessary to place round it a metallic screen (or what is quite efficient, a metallic grating or network) connected with the earth.

Principle of an Electrical Screen.

Tube of Force.

Consider a small element of area ds of an equipotential surface, and imagine lines of force to be drawn through every point in its periphery so as to form a tubular surface. Such a surface is called a *tube of force.* Let R be the resultant electric force at the element ds and let Rds be taken as unity, the tube is then a *unit tube.* Imagine any finite area of the equipotential surface to be divided into elements such that the tube for each is a unit tube, and let n be their number so that $\Sigma Rds = n$; then n is the number of tubes of force, or, as it is usually put, "the number of *lines* of force" which cross the area. It is to be remembered that what is called a line of force in the phrase "number of lines of force" is a unit tube. We shall not however use the term in this sense, but in the sense defined above (p. 6), which has no reference to intensity of force.

Using N in the same sense as before, and considering the projection of ds on an equipotential surface at the element, we see that Nds is really the same thing as the number of tubes of force which cross ds, and as N is to be taken positive where the lines leave the surface, and negative where they enter it, we see that the excess of the number of tubes of force which leave the surface over the number which enter it, that is the electric induction over the surface, is equal to 4π times the algebraical sum of the electricity within the surface.

Let ds and ds' be two normal sections of a tube of force, and let F and F' be the force at the two sections, then the surface integral of electric induction for the portion of the tube between

the sections is $Fds - F'ds'$, and if there be no electricity within this part of the tube

$$Fds = F'ds'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

that is, the product of the electric force and the cross-sectional area of the tube is everywhere constant if the tube contain no electricity.

Tube of Force passing through an Electrified Surface.

If however the tube pass through an electrified surface, then we may suppose the tube so thin that the normal at every point of the intercepted element of the surface is in the same direction, and consider the very short portion of the tube bounded by two cross-sections parallel to the element, one just within and the other just without the surface. If ds be the area of the element intercepted on the surface by the tube, σ the density there of the distribution, and θ the angle which the normal to the element on the positive side makes with the direction there of the resultant force, we have, taking the surface integral, which consists only of the end portions,

$$(F - F') \cos \theta ds = 4\pi\sigma ds,$$

$$(F - F') \cos \theta = 4\pi\sigma \quad . \quad . \quad . \quad . \quad . \quad (25)$$

that is, the normal components of electric force at two neighbouring points on a line of force, but on opposite sides of the surface, differ by 4π times the electric surface density where the line cuts the surface.

If we draw normals ν, ν' *outwards* from the two ends of the portion of tube here considered, and V_1, V' be the respective potentials at the two sides of the surface, we have

$$F \cos \theta = -dV/d\nu, \;\; F' \cos \theta = dV'/d\nu',$$

and, therefore, equation (25) becomes

$$\frac{dV}{d\nu} + \frac{dV'}{d\nu'} + 4\pi\sigma = 0 \quad . \quad . \quad . \quad . \quad . \quad (26)$$

This is the form which the characteristic equation of the potential takes at a surface at which the electric force is discontinuous, and it shows that to the discontinuity there corresponds a certain determinate density of electric distribution on the surface. Since the potential is constant within the substance of a conductor on which electricity is in equilibrium, a tube of force must be considered as terminating just within the electrified surface. Hence the surface integral for a tube of force extending between and terminating in the substance of

two conductors, is equal to zero. The total quantity of electricity within the tube is therefore also zero, and hence if ds, ds' be the elements intercepted by the tube on the two surfaces, and σ, σ' the corresponding surface densities, we have

$$\sigma ds + \sigma' ds' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Force at Surface of a Conductor.

If we consider a tube of force terminating at one end just within the electrified surface of a conductor, and at the other end just outside the surface, we have for the end within the surface $F' = (-dV'/d\nu') = 0$, and therefore

$$F = -dV/d\nu = 4\pi\sigma. \quad . \quad . \quad . \quad . \quad (26\ bis)$$

Hence the density at any element of an electrified surface is $F/4\pi$, where F is the force at an external point infinitely near the element.

Surface Distribution Replacing Internal System;

From the result obtained above (pp. 25, 26) for a closed conductor, containing an electric system insulated within it, it follows that, whether or not there be an external electric system, the electrification of the inner surface reversed in sign, would produce exactly the same potential at the conductor and all external points as is due to the internal system. But by (26 *bis*) the density at any point of the inner surface is $-F/4\pi$, where F is the internal force at the point in the *outward* direction. The density of the distribution which on a surface coinciding with the inner surface of the conductor would replace for external points the internal system is therefore $F/4\pi$.

Case of Conductor Coinciding with Equipotential Surface.

Suppose an infinitely thin insulated conductor made coincident with an equipotential surface of an electric system whether wholly or partly internal, and the internal system replaced by that distribution over the conductor which does not alter the potential at the surface or at any external point. The force at any

Case of Conductor Coinciding with Equipotential Surface.

point just outside the surface has its former value F, and the electric density there is, by (26 *bis*), $F/4\pi$. This (pp. 69, 76) is the only distribution which fulfils the prescribed conditions, and since the conductor is all at one potential in the equilibrium distribution, the total charge is, as we have seen, equal to the charge of the internal system.

Since this surface distribution is that of equilibrium it is that which the conductor would take if insulated without charge in presence of the actual electric system, and as we have just seen it is identical with the infinitely nearly coincident distribution on the interior surface reversed in sign. Hence no change in potential or force at any point external or internal is produced by making an infinitely thin conducting shell insulated without charge coincident with the equipotential surface.

Section II.

POTENTIAL ENERGY OF AN ELECTRIC SYSTEM.

GENERAL PROPOSITIONS REGARDING A SYSTEM OF CONDUCTORS.

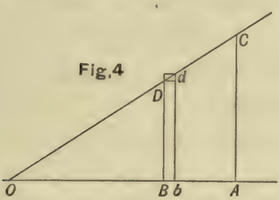
Electric Energy.

The potential energy of any electric system can depend only on the state of the system. Now the principle of superposition stated above gives a method of calculating the work spent in charging the system by a particular process, and therefore also the potential

energy of the system, provided we can assume its equality to the work so spent. But by the principle of the Conservation of Energy the work spent in bringing a material system from one state to another, in any manner whatever, is the equivalent of the excess of the energy of the system in the latter state over its energy in the first. Hence we may assume that the work spent in electrifying the system by any series of charges whatever is the equivalent of the electric energy stored up in the system. Electric Energy.

Let the system proceed from zero electrification to the final state by infinitesimal steps, each such that the Calculation of Electric Energy of any System.

Fig. 4



relative electrifications of all the parts of the system are the same as in the final state. By the principle of superposition the increments of the charges positive or negative of the various parts will be proportional to the changes of potential which take place, that is, both the electrification and the potential of every part of the system change uniformly with the time if one does so. Hence if we represent the final charge of any element by a straight line OB, and the final potential by the ordinate AC, the potential of the element corresponding to any intermediate charge OB will be given by the ordinate BD drawn from B and meeting OC in D. Now the

Calculation of Electric Energy.

work spent in the process of charging described, in giving a charge q to any element at which the potential is V_1 before the charge is given and V_2 immediately after, is greater than $V_1 q$ and less than $V_2 q$; that is, if Bb represent q, the work done in bringing q to the element lies in numerical value between those of the rectangular areas bD and Bd. But these two rectangles differ by the rectangle Dd, which is very small in comparison with either, when Bb is a small fraction of OB. Hence the whole work spent in charging the element is, if its final potential be V and charge Q, numerically equal to the area of the triangle OAC or $\frac{1}{2}VQ$. If E be the energy of the whole system we have

$$E = \tfrac{1}{2}\Sigma VQ \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

Expression for Potential Energy of Electric System.

where Σ denotes summation of the products VQ taken for all the elements.

In the case of conductors whose potentials are V_1, V_2, and charges Q_1, Q_2, &c., we have

$$E = \tfrac{1}{2}(V_1 Q_1 + V_2 Q_2 \text{ \&c.}) = \tfrac{1}{2}\Sigma VQ \quad . \quad . \quad (29)$$

where V denotes the potential of any conductor and Q its charge.

For any system of surface distributions whether in whole or in part on conductors or not, at any element ds of which the electric surface density is σ, we have for the energy

$$E = \tfrac{1}{2}\iint V\sigma \, ds \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where the integration is extended over all the surfaces. Similarly if there be a volume distribution with potential varying from point to point we have for the energy of an element $dx\,dy\,dz$ at which the potential is V and density ρ the value $\frac{1}{2}V\rho\,dx\,dy\,dz$. Hence for the total energy in the most general case we have

$$E = \tfrac{1}{2}\iint V\sigma \, ds + \tfrac{1}{2}\iiint V\rho \, dx\,dy\,dz \quad . \quad . \quad . \quad (31)$$

the surface integral being extended over all the electrified bodies and the volume integrals over all the spaces occupied by electricity. If instead of σ we write its value $-\dfrac{1}{4\pi}\dfrac{dV}{d\nu}$, and for ρ its value $\dfrac{1}{4\pi}\nabla^2 V$, we get

Electrified System—First Expression for Energy.

$$E = -\frac{1}{8\pi}\iint V\frac{dV}{d\nu}ds - \frac{1}{8\pi}\iiint V\nabla^2 V\,dx\,dy\,dz \quad \ldots (32)$$

It is easy to obtain another expression for the electric energy. Imagine two equipotential surfaces of the actual system at a very short distance apart, and let the electric force at any element ds of the inner surface be F, and the distance between the surfaces measured along a line of force $d\nu$. Now imagine electricity to be gradually distributed over the inner surface so as to produce finally the resultant force F at each point just outside the surface, and the charge on each element to be brought along a line of force to that element from the outer surface, and so that the distribution on the surface has always the same relative density. If f be the electric force at ds due to the distribution on the surface, at any stage in its building up in this manner, the work done in bringing a small quantity of electricity dq along $d\nu$ to ds against f, is $dq\,.\,fd\nu$. By this transfer the electric force has been changed from f to $f + df$, and the surface density therefore increased by $df/4\pi$. But $dq = ds\,.\,df/4\pi$, hence the work done is $\dfrac{1}{4\pi}d\nu.ds.fdf$. If the inner surface is originally uncharged f varies from 0 to F, and the work done over the surface is

$$\frac{1}{4\pi}d\nu\iint ds\int_0^F fdf,$$

or

$$\frac{1}{8\pi}d\nu\iint F^2\,ds \quad \ldots\ldots \quad (33)$$

Second Expression for Energy.

and obviously, by adding the values of this integral for successive equipotential surfaces we shall obtain the whole electrical energy of the system. We have therefore

$$F = \frac{1}{8\pi}\iiint F^2\,dx\,dy\,dz \quad \ldots\ldots \quad (33\ bis)$$

the integration being extended throughout all space. This expression gives the value of the total electrical energy for any

Second Expression for Energy.

distribution whatever. In the case of a system of electrified conductors the integration need not of course be extended to the space occupied by the substance of any conductor or to the space within any conductor if it contain no electricity, as in every such space the value of F is zero. Using (20) we get by (32) and (33)

$$\left.\begin{aligned} E &= \frac{1}{8\pi}\iiint \left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\} dx\,dy\,dz \\ &= -\frac{1}{8\pi}\iint V\frac{dV}{dv}\,ds - \frac{1}{8\pi}\iiint V\nabla^2 V\,dx\,dy\,dz, \end{aligned}\right\} \quad (34)$$

the energy equations usually deduced by Green's general theorem.

The Electric Field as Seat of the Energy.

The first expression in (34) suggests the energy as having its seat in the medium occupying the field; and, by the proof given on p. 33, unit tubes of force intersected by successive equipotential surfaces, drawn at unit differences of potential, are divided into spaces each of which gives to the sum half a unit of energy. Maxwell has called these spaces *unit cells*.

As an interesting example of these equations we may find the energy spent in bringing together into a uniform sphere, from a state of uniform diffusion throughout infinite space, matter, the parts of which repel one another according to the law stated on p. 2; or, which is the same thing, the energy gained by allowing matter attracting according to the same law to come together thus from the nebular state.

Change of Potential Energy in Formation of Uniform Sphere from Nebula.

Let the radius of the sphere be r and its density ρ. The first term of the expression on the right-hand side of (34) is here zero, and the energy is $\frac{1}{2}\iiint V\rho dxdydz$, the integral being taken throughout the sphere. If we consider a spherical surface of radius x we see that this expression may be put in the form $2\pi\rho\int_0^r Vx^2dx$ where V is the potential at any point on the surface. But by equation (20) $V = 2\pi\rho(r^2 - \frac{1}{3}x^2)$; hence

$$E = 4\pi^2\rho^2\int_0^r x^2(r^2 - \tfrac{1}{3}x^2)dx = \tfrac{3}{5}\frac{M^2}{r}$$

where $M\left(= \dfrac{4\pi\rho}{3}r^3\right)$ is the mass of the sphere.

The same result may be obtained from the equation

$$E = \frac{1}{8\pi}\iiint F^2\,dxdydz.$$

The integral here is taken through all space. We divide it into two parts, (1) that due to space external to the sphere, and (2) that due to the space contained within the spherical surface, and evaluate these separately.

Change of Potential Energy in Formation of Uniform Sphere from Nebula.

In the first case, at any point at distance x from the centre of the sphere, $F^2 = M^2/x^4$, and therefore the first part of the integral is

$$\frac{M^2}{2}\int_r^\infty \frac{dx}{x^2} = \frac{1}{2}\frac{M^2}{r}.$$

In the second case, by equation (21), $F^2 = 16/9 \,.\, \pi^2\rho^2x^2$ at any point of an internal concentric spherical surface of radius x. For the second part of the integral therefore we have

$$\frac{8\pi^2\rho^2}{9}\int_0^r x^4dx = \frac{8\pi^2\rho^2}{9\,.\,5}r^5 = \frac{1}{10}\frac{M^2}{r}.$$

Adding these two parts, we get

$$E = \frac{3}{5}\frac{M^2}{r}$$

the same result as before.

Calculation of Tension at an Electrified Surface.

If we denote by P the force exerted on an element ds of the electrified surface of a conductor by the whole electrified system, we have for the work done in transferring the charge on the element a distance $d\nu$ along lines of force to the corresponding element of an adjacent equipotential surface, the value $Pd\nu\,.\,ds$. But by equation (33) this is $\frac{1}{8\pi}\,d\nu\,.\,F^2ds$. Hence

$$P = \frac{1}{8\pi}F^2 = 2\pi\sigma^2 \quad . \quad . \quad . \quad . \quad (35)$$

Electric Diminution of Air Pressure in a Soap Bubble.

This is the outward force exerted by the element ds of the conductor on the medium, and measures therefore the reaction of the medium on the element. For example every element of an electrified soap-bubble exerts an outward pressure on the surrounding air equal to $2\pi\sigma^2$ per unit of area, which may be regarded

Electric Diminution of Air Pressure in a Soap Bubble.

as a diminution of the air pressure on the outer surface; or, if the bubble is spherical and of radius r, the surface tension of the film is by capillary theory apparently diminished by the amount $\frac{1}{2}\pi\sigma^2 r$.

The outward pressure P on the medium is what has been called the "electric tension" at a point on an electrified surface, and is the true measure of the tendency to discharge. Its proportionality to σ^2 explains the so-called power of points.

Additional Proof of Expression for Tension at an Electrified Surface.

The value of P may be obtained otherwise thus. We may regard the surface distribution as a limiting case of a volume distribution of density ρ, and take the axis of z along the normal to the surface from the inside to the outside of the stratum. Then since we may consider the portion of the surface surrounding the normal as a part of a uniform plane distribution, the electric force does not vary along the plane, and Laplace's equation reduces to $-\frac{1}{4\pi}\frac{d^2V}{dz^2} = \rho$. But if ρ is finite however great, we may write

$$P = \int -\frac{dV}{dz}\rho\, dz = \frac{1}{4\pi}\int \frac{dV}{dz}\frac{d^2V}{dz^2}dz,$$

and integrate from the inside to the outside of the stratum. Hence since $F\left(= \frac{dV}{dz}\right)$ is zero on the inside of the stratum we have

$$P = \frac{1}{8\pi}F^2 = 2\pi\sigma^2,$$

Additional Proof of Second Expression for Energy.

where F is the resultant electric force just outside the surface.

This equation might now be applied to form equation (33), and hence to give at once the expression (33 *bis*) for the electric energy.

Considering two electric distributions A, B, in the same electric field, let the potential produced by B at any point P in A be V', and that produced by A at any

point P be V, and let dq be an element of electricity at P, dq' an element at P'. We have then the relation

Mutual Potential Energy of two Electric Systems.

$$\Sigma_A V'dq = \Sigma_B V dq' \quad . \quad . \quad . \quad . \quad (36)$$

where Σ_A denotes summation for every element of A, and Σ_B summation for every element of B. For the expression on the left is plainly the work which would be done if the distribution on B remaining unchanged, the system A were removed to an infinite distance, and that on the right the work which would be done if, the distribution in A remaining unchanged, the system B were removed to an infinite distance; and it is plain that the same amount of work must be done in both cases.

Each of the expressions is, in fact, the mutual potential energy of the two systems.

The relation may be thus proved analytically. Since $V' = \Sigma_B \frac{dq'}{r}$, and $V = \Sigma_A \frac{dq}{r}$, we have

$$\Sigma_A V'dq = \Sigma_A dq \Sigma_B \frac{dq'}{r} = \Sigma_B dq' \Sigma_A \frac{dq}{r} = \Sigma_B V dq'.$$

Reciprocal Relation of two States of same System.

There is nothing to prevent two equilibrium states of a system of conductors, C_1, C_2, &c., from being taken as A and B. Then if Q_1, Q_2, &c., Q'_1, Q'_2, &c., be the charges, we have, since V and V' are constants for any one conductor,

$$V'_1 Q_1 + V'_2 Q_2 + \text{\&c.} = V_1 Q'_1 + V_2 Q'_2 + \text{\&c.} \quad (37)$$

This reciprocal relation can be proved for the case of one and the same system of conductors in the following simple manner. We may suppose the change from the

Reciprocal Relation of two States of same System.

state A to the state B to take place simultaneously for all the conductors, in such a way that the ratio of the change produced in the charge of a conductor to the total change from one state to the other, has the same value in each case. Since each increase or decrease of the charge of any one conductor produces a change in the potential of each conductor proportional to that increase or decrease, and these changes can be superimposed, it is plain that equal proportionate changes in the charges of all the conductors will produce equal proportionate changes in their potentials. Hence if OA (fig. 5) represent the initial charge of any one

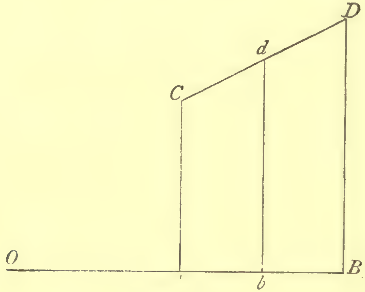


FIG. 5.

conductor, and AC its corresponding potential, OB its final charge, and BD its final potential, Ob any intermediate charge and bd the corresponding potential, the point d will lie on the straight line CD. The work done in altering the charge of the conductor is equal to the area of the trapezoid, or $\frac{1}{2}(AC + BD)\,(OB - OA)$, that is $\frac{1}{2}(V + V')\,(Q' - Q)$. The work done in bringing all the conductors from the state A to the state B, is

$\frac{1}{2}\Sigma(V + V')(Q' - Q)$. But the energy of the system in the initial state is $\frac{1}{2}\Sigma VQ$ and in the final state $\frac{1}{2}\Sigma V'Q'$. We have therefore

Reciprocal Relation of two States of same System.

$$\Sigma V'Q' - \Sigma VQ = \Sigma(V + V')(Q' - Q)$$

that is
$$\Sigma VQ' = \Sigma V'Q,$$

or
$$\Sigma(V + V')(Q' - Q) = \Sigma(V' - V)(Q + Q').$$

Problem of a System of Conductors.

Let the potentials and charges of a system of n conductors C_1, C_2, &c., C_n in the same electric field be V_1, V_2, &c., V_n; Q_1, Q_2, Q_3, &c., Q_n respectively. Since the potential any point is $\iint \frac{\sigma ds}{r}$, where σ is the density at an element ds of the system and r is the distance from ds to the point in question, and the integration is extended over the system, the portion of the potential contributed by each conductor varies directly as the charge of the conductor. The potential of any conductor is therefore a linear function of the charges of the conductors—that is, we have a series of equations of the form

$$\left.\begin{array}{l} V_1 = p_{11}Q_1 + p_{21}Q_2 + \text{\&c.}, + p_{n1}Q_n \\ V_2 = p_{12}Q_1 + p_{22}Q_2 + \text{\&c.}, + p_{n2}Q_n \\ \qquad \text{\&c.} \qquad\qquad\qquad \text{\&c.} \\ V_n = p_{1n}Q_1 + p_{2n}Q_2 + \text{\&c.}, + p_{nn}Q_n \end{array}\right\} \quad . \quad (38)$$

Coefficients of Potential.

where p_{11}, p_{22}, &c., p_{21}, p_{12}, &c., are coefficients which depend only on the relative positions of the conductors. They are called *coefficients of potential.* The first suffix of each coefficient refers to the conductor to which the charge belongs, the second to that whose potential

Coefficients of Potential.

is given by the equation in which the coefficient occurs.

The physical meaning of the coefficients p_{11}, p_{22}, p_{33}, &c., in which the suffixes are alike, is easily seen. Let any conductor C_k be charged with unit quantity of electricity, and all the other conductors be without charge. We get in that case

$$V_k = p_{kk}$$

Reciprocal Relations of Potentials,

that is p_{kk} is the potential produced in C_k by unit charge on C_k itself, when all other conductors are without charge.

Again, to determine the physical meaning of the other coefficients, let C_k have unit charge and all the others zero charge. For the potential of C_j we have

$$V_j = p_{kj}$$

that is, p_{kj} is the potential produced at C_j by unit charge on C_k, when all the other conductors are without charge.

Now (1) let C_j have unit charge, and each of the other conductors zero charge, and (2) let C_k have unit charge and each of the others zero charge. The potential of C_k in case (1) is p_{jk}; and the potential of C_j in (2) is p_{kj}. Applying the theorem of (36) above we get at once the reciprocal relation

$$p_{jk} = p_{kj}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

(1) All Conductors Insulated, only one Charged.

that is, the potential in C_j produced by unit charge on C_k is the same as the potential produced at C_j by unit charge on C_k, if all the other conductors be insulated and without charge.

(2) Two or more Conductors Insulated, only one of these Charged

Again the potential at C_j produced by unit charge on C_k is the same as the potential produced at C_k by unit charge on C_j, if some (or all) of the other conductors be maintained at potential zero, and the rest (if any) with C_k, be insulated without charge. We may evidently consider the former conductors as one conductor C_s. If then C_j have unit charge while C_k is insulated, and Q_s be the charge of C_s, we have

$$V_k = p_{jk} + p_{sk}Q_s$$
$$V_s = p_{js} + p_{ss}Q_s = 0$$

therefore

$$V_k = p_{jk} - \frac{p_{sk}\,p_{js}}{p_{ss}} \quad . \quad . \quad . \quad . \quad . \quad (40)$$

Now, let C_k have unit charge while C_j is insulated and without charge, and let Q'_s be the charge of C_s, then

$$V_j = p_{kj} + p_{sj}Q'_s$$
$$V_s = p_{ks} + p_{ss}Q'_s = 0$$

and therefore

$$V_j = p_{kj} - \frac{p_{sj}\,p_{ks}}{p_{ss}} \quad . \quad . \quad . \quad . \quad . \quad (41)$$

But by the relation already proved $p_{kj} = p_{jk}$; $p_{sj} = p_{js}$, and $p_{sk} = p_{ks}$; hence by (38) and (39)

$$V_j = V_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

the theorem stated above.

Green's Theorem of Reciprocal Potentials.

The second term in the expressions for V_j and V_k is the potential produced at the corresponding conductor by the induced electrification in C_s, and it is the same in both cases. This is a theorem first given by Green.

There are $\frac{1}{2}n(n-1)$ equations of the form $p_{jk} = p_{kj}$, one

Green's Theorem of Reciprocal Potentials.

for each pair of the $n(n-1)$ coefficients which have different suffixes j, k: hence there are really only $\frac{1}{2}n(n-1)$ coefficients, one for each different pair of conductors which can be formed from the given system.

By solving equations (37) above for Q_1, Q_2, &c., Q_n we get a system of n equations of the form

$$\left.\begin{array}{l} Q_1 = q_{11}V_1 + q_{12}V_2 + \text{\&c.} + q_{1n}V_n \\ Q_2 = q_{21}V_1 + q_{22}V_2 + \text{\&c.} + q_{2n}V_n \\ \quad\text{\&c.} \qquad\qquad \text{\&c.} \\ Q_n = q_{n1}V_1 + q_{n2}V_2 + \text{\&c.} + q_{nn}V_n \end{array}\right\} \quad .. \quad (43)$$

Capacity of a Conductor.

where q_{11}, q_{22}, &c., q_{12}, q_{21}, &c., are coefficients, which, like those of potential, depend only on the relative positions of the conductors. The meaning of any coefficient q_{kk}, of which the two suffixes are alike, can be obtained by supposing the potentials of all the conductors, except C_k, zero, and C_k to be at unit potential. The equation for the conductor C_k is then

$$Q_k = q_{kk},$$

—that is, q_{kk} is the quantity of electricity required to charge C_k to unit potential, all the other conductors being kept at potential zero. The coefficients of this form, q_{11}, q_{22}, q_{33}, &c., q_{nn} are called the *capacities* of the respective conductors C_1, C_2, &c., C_n in the given system.

Coefficients of Induction.

To find the meaning of the coefficients of the form q_{jk}, in which the suffixes are not alike, let C_k as before be kept at unit potential and all the others at potential zero. The equation for C_j is then plainly

$$Q_j = q_{jk}$$

—that is, q_{jk} is the quantity of electricity on C_j when, along with all the other conductors except C_k, it is at zero potential and C_k is at unit potential. The coefficients q_{12}, q_{13}, q_{23}, &c. of this form are called *coefficients of induction.*

Coefficients of Induction.

A reciprocal relation

$$q_{jk} = q_{kj} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

exists for these coefficients also. The proof is precisely the same as that given above for the potential coefficients, except that "potential" is to be read for "charge" and "charge" for "potential."

Conditions fulfilled by Coefficients.

Every coefficient of the form p_{kk} is positive, and every coefficient of the form p_{jk} is intermediate in value between zero and p_{kk} or p_{jj}. For, let C_j be charged with a unit of positive electricity and all the other conductors be insulated and uncharged, the electric induction over C_j, or over a closed surface surrounding it, is 4π, and the potential of the conductor is positive. The electric induction over *any other* conductor C_k is zero. As many unit tubes of force terminate in C_k as originate in it, and therefore the potential must at some places increase outwards, at others diminish outwards from C_k—that is, there must be a conductor in the field which has a higher potential than C_k has, while the potential of C_k must be greater than zero. The conductor of highest potential must be C_j, which is the only conductor in the field whose coefficient is not of the form p_{jk}; hence p_{jk} is not greater than p_{kk}; and similarly it can be shown that it is not greater than p_{jj}. If any conductor C_l be inclosed within C_k, it will

have the same potential as C_k, and in that case therefore $p_{jl} = p_{jk}$.

Capacities of Conductors all Positive.

The capacities q_{11}, q_{22}, &c. of the conductors are all positive. For suppose as before C_k at unit potential and all the other conductors at zero potential; then q_{kk} is the charge of C_k. The potential diminishes in every direction outwards from C_k, and therefore the surface integral of electric induction is positive, that is $4\pi q_{kk}$ is positive. The electrification of C_k is everywhere positive.

Induction Coefficients all Negative.

The coefficients of induction q_{jk} are all negative. For suppose C_k charged as before. Since C_j is at zero potential and all other conductors except C_k are also at zero potential, the potential cannot diminish in any direction outwards from C_j and must increase towards C_k. Hence the electric induction over C_j, that is $4\pi q_{jk}$, is negative. If C_j be inclosed within another conductor, q_{jk} is of course zero.

The sum of the coefficients of induction of the system for any one conductor cannot be greater than the capacity of that conductor. The electric state of the system remaining the same, let a closed surface be described inclosing the whole. The potential cannot increase in any direction outwards across this surface. Otherwise, since the potential is zero at an infinite distance, a place of maximum potential would exist in free space outside the conductor. It may, however, diminish outwards: therefore the electric induction over the closed surface cannot be negative. Hence $q_{1k} + q_{2k} +$ &c. cannot be greater than q_{kk}. When the other conductors completely inclose C_k,

$$q_{1k} + q_{2k} + \text{\&c.} = q_{kk}.$$

Electric Field explored by carrying Small Charged Ball round Conductor at Zero Potential.

The first reciprocal relation established above, equation (39), gives a convenient means of exploring the electric field due to a charged conductor of any form.[1] One electrode of a delicate electrometer (Chap. IV.) is connected with the conductor, supposed insulated and uncharged, and the other electrode is connected to the earth. Then a small charged sphere carried by an insulating handle is placed with its centre at any point of the field, and the electrodes of the electrometer connected for an instant. The conductor is thus reduced to potential zero. The sphere is next moved from point to point in the field, and the positions noted for which the electrometer shows no deflection. These positions lie on an equipotential surface of the conductor. For by (39) the potential at the conductor due to the electrification of the sphere is equal to the potential which would be produced at the sphere by a charge on the conductor equal to that on the sphere, and this part of the potential is the same for all positions of the sphere for which there is zero deflection. By the principle of superposition this must be an equipotential surface for all charges of the conductor.

The convenience of the method consists in the zero potential of the conductor, which therefore does not lose or gain electricity, while the exploring sphere, which can be insulated so as to lose its charge only with extreme slowness, is changed in position.

[1] Maxwell, *Elementary Treatise on Electricity and Magnetism*, p. 43.

Section III.

ELECTROSTATIC CAPACITY. ELECTRIC DISTRIBUTION ON ELLIPSOIDS. ELECTROSTATIC CAPACITY IN SIMPLE CASES.

Electrostatic Capacity.

The capacity of an insulated conductor is, as we have seen above (p. 42), the quantity of electricity required to charge the conductor to unit potential, when all the other conductors in the field are maintained at potential zero. Hence if the potential of such a conductor be V, the corresponding charge Q, and the electrostatic capacity C, we have

$$C = \frac{Q}{V}. \quad \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Capacity of a Spherical Conductor.

The capacity of a conductor depends on its position relatively to other conductors, as well as on its form and dimensions, and its determination in any given case involves finding the distribution of electricity upon it in the given circumstances when all other conductors in the field are maintained at potential zero. The electrostatic capacity is easily found in the following cases, which will be useful in what follows.

1. A Spherical Conductor at an infinite distance from all other conductors.

Let r be the radius of the conductor, q its charge. The potential at the surface is $\frac{q}{r}$ and therefore

$$C = q \Big/ \frac{q}{r} = r \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

or the electrostatic capacity is numerically equal to the radius of the sphere.

Capacity of a Spherical Conductor.

If r is 1, C is also 1; hence the unit of electrostatic capacity is the capacity of a sphere of unit radius at an infinite distance from all other conductors.

Capacity of an Ellipsoidal Conductor.

2. An Ellipsoidal Conductor at an infinite distance from all other conductors.

Density of Equilibrium Distribution.

The density of the distribution at each point is proportional to the thickness there of a thin elliptic homœoid,[1] the inner surface of which coincides with the given surface. For there can be no force within the ellipsoidal conductor, and it is easy to show that this condition is fulfilled by the distribution stated, which therefore is the only possible distribution. Such a thin shell may be considered as formed by subjecting a thin uniform spherical shell to homogeneous strain, that is, straining it so that pairs of points initially equidistant and in parallel lines remain equidistant and in parallel lines. Let S and S' be the inner and outer surfaces of such a shell supposed composed of attracting matter, and let lines forming a small cone be drawn from any point O in the interior. Let $pp'q'q$, $rr's's$ be the portions of the homœoid intercepted by the cone. The masses of these portions are the corresponding unstrained masses in the spherical shell, and the ratio of the distances Op, Or, has not been altered by the strain. Hence (p. 14) the attractions of the frustums $pp'q'q$, $rr's's$

Attraction of an Elliptic Homœoid at an internal point.

[1] Thomson and Tait (*Natural Philosophy*, vol. i. part 2, § 494 g, footnote) call a shell bounded by two similar, similarly situated and concentric surfaces, a *homœoid*. When the surfaces are ellipsoids the shell is an *elliptic homœoid*.

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Attraction of an Elliptic Homœoid at an internal point.

on a particle at O are equal and opposite. Dividing in a similar manner the whole surface into pairs of opposite elements by cones drawn from O we can show that the resultant force at any internal point O is zero.

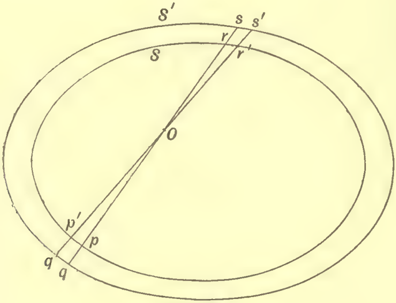

Fig. 6

Equivalent Law of Density.

The potential is therefore constant throughout the interior. It is evident therefore that an ellipsoidal conductor charged so that its electric density at any point is proportional to the thickness of a thin elliptic homœoid, having the conductor for its inner surface, exerts no force at any internal point, and hence that this must be the actual distribution on a conducting ellipsoid in equilibrium. Since the density varies as the thickness of the material homœoid, it follows that its values at different points are proportional to the lengths of the perpendiculars let fall from the centre on the tangent planes at the respective points.

For let $x + dx$, $y + dy$, $z + dz$, be the point in which the outer surface is cut by a normal drawn to the inner at the point x, y, z. These points respectively satisfy the equations—

Equivalent Law of Density.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$\frac{(x + dx)^2}{a^2(1 + \nu)} + \frac{(y + dy)^2}{b^2(1 + \nu)} + \frac{(z + dz)^2}{c^2(1 + \nu)} = 1.$$

If p be the perpendicular from the centre on the tangent plane at x, y, z, and r the portion of the normal intercepted between the surfaces, we have

$$\frac{r}{p} = \frac{x}{a^2}\,dx + \frac{y}{b^2}\,dy + \frac{z}{c^2}\,dz \quad . \quad . \quad . \quad . \quad (47)$$

But by the equations of the surfaces we have

$$\frac{(x + dx)^2}{a^2(1 + \nu)} - \frac{x^2}{a^2} + \frac{(y + dy)^2}{b^2(1 + \nu)} - \frac{y^2}{b^2} + \frac{(z + dz)^2}{c^2(1 + \nu)} - \frac{z^2}{c^2} = 0.$$

Since dx, dy, dz, ν are small this is

$$2\left(\frac{x}{a^2}\,dx + \frac{y}{b^2}\,dy + \frac{z}{c^2}\,dz\right) - \nu = 0,$$

or by (47)

$$\tau = \tfrac{1}{2}p\nu \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

that is, the thickness of the homœoid varies from point to point directly as the length of the perpendicular from the centre on the tangent plane.

Equipotential Surfaces Confocal Ellipsoids.

The electric force at a point infinitely near the surface of the ellipsoidal conductor has the value $4\pi\sigma$. Now the rate of variation of the potential with distance outwards from the surface, or the force, is inversely proportional to the distance between the surface of the conductor and an equipotential surface infinitely near it. Hence the distance is inversely proportional to σ, that is to the thickness of the material homœoid; or, which is the same, to the length of the perpendicular from the centre to the tangent plane at the point considered.

Let x, y, z be a point on the surface of the conductor and τ the distance from x, y, z along the normal to a point $x + dx$, $x + dy$,

Equipotential Surfaces Confocal Ellipsoids.

$z + dz$ on a neighbouring equipotential surface. We have as before

$$\tau = p\left(\frac{xdx}{a^2} + \frac{ydy}{b^2} + \frac{zdz}{c^2}\right).$$

But by the considerations just stated $\tau = \frac{1}{2}\lambda/p$, where λ is a small multiplier constant over the surface. Hence

$$\frac{xdx}{a^2} + \frac{ydy}{b^2} + \frac{zdz}{c^2} = \frac{1}{2}\frac{\lambda}{p^2}$$

$$= \frac{1}{2}\lambda\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right).$$

Therefore

$$dx = \frac{1}{2}\frac{\lambda x}{a^2},\quad dy = \frac{1}{2}\frac{\lambda y}{b^2},\quad dz = \frac{1}{2}\frac{\lambda z}{c^2},$$

or

$$\frac{2xdx}{\lambda} = \frac{x^2}{a^2},\quad \frac{2ydy}{b^2} = \frac{y^2}{b^2},\quad \frac{2zdz}{c^2} = \frac{z^2}{c^2}.$$

Hence to small quantities of the second order

$$\frac{(x + dx)^2}{a^2 + \lambda} + \frac{(y + dy)^2}{b^2 + \lambda} + \frac{(z + dz)^2}{c^2 + \lambda} = 1 \quad . \quad . \quad (49)$$

The equipotential surface infinitely near the conductor is therefore a confocal ellipsoid, and we see in the same way that the successive equipotential surfaces are ellipsoids confocal with the given ellipsoid.

Proof for Elliptic Homœoids of Maclaurin's Theorem.

By distributing the whole quantity of attracting (or repelling) matter over any equipotential surface, so that the potential may be constant within the surface, and have the same value at every external point as in the actual case, we form a thin elliptic homœoid having its inner surface coincident with the surface. Hence the attractions of any two thin confocal elliptic homœoids of the same mass on a point external to both are the same. It follows that any two elliptic homœoids of finite

thickness, the inner and outer surfaces of which are confocal and which have the same mass, exert the same force at all points external to both. For it is possible to divide each homœoid into the same number of thin homœoids which are one by one (proceeding from within outwards) confocal with and of equal mass to those of the other, and exert therefore the same attraction at all external points.

Proof for Elliptic Homœoids of Maclaurin's Theorem.

Further, if the hollow space within one shell be infinitely small, that within the other is also infinitely small, and we see that two confocal ellipsoids of equal mass exert the same attraction at all points external to both. This is Maclaurin's Theorem of Attraction of Ellipsoids.* The mode of deriving it from the theorem of equivalence of confocal homœoids of equal mass is due to Chasles.

Proof of Maclaurin's Theorem.

We can now find the potential of the ellipsoidal conductor for any given charge. To find the force at any point P, let $\alpha\ (= \sqrt{a^2+\lambda})$, $\beta\ (= \sqrt{b^2+\lambda})$, $\gamma\ (= \sqrt{c^2+\lambda})$ be, (49) above, the axes of an equipotential surface passing through P, and let ds be a small element of the surface including P. The density, at ds, of the equivalent electric distribution over the surface, may be taken as numerically equal to the thickness there of an elliptic homœoid with axes, α, β, γ; $\alpha(1+\delta\nu)^{\frac{1}{2}}, \beta(1+\delta\nu)^{\frac{1}{2}}, \gamma(1+\delta\nu)^{\frac{1}{2}}$. If ϖ be the length of a perpendicular from the centre to the tangent plane at P, we have, by (48) above, for the thickness $\frac{1}{2}\varpi\delta\nu$ and for the force $2\pi\varpi\delta\nu$. Let p and n be the corresponding quantities for an element ds of the given conductor and we have two expressions for the total charge

Potential of an Ellipsoidal Conductor at External Point.

$$\frac{\delta n}{2}\iint pds = \frac{\delta\nu}{2}\iint \varpi ds,$$

or, since $\iint pds = 4\pi abc, \quad \iint \varpi d\sigma = 4\pi\alpha\beta\gamma$

$$2\pi abc\delta n = 2\pi\alpha\beta\gamma\delta\nu.$$

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* See Thomson and Tait's *Nat. Phil.*, vol. i. part ii. §§ 494, 522.

Hence the force is $2\pi\,\dfrac{abc}{\alpha\beta\gamma}\,\varpi\delta n$. Now, for the distance between the equipotential surface, and one infinitely near it we have the value $d\lambda/2\varpi$; hence the work done in carrying a unit of positive electricity along a line of force from one to the other is $\pi\delta n\,\dfrac{abc}{\alpha\beta\gamma}\,d\lambda$. The potential V is therefore given by the equation,

$$V = \pi\delta nabc \int_o^\infty \frac{d\lambda}{(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}}} \quad . \quad . \quad (50)$$

Capacity of Ellipsoidal Conductor.

Dividing the value of the charge, $2\pi abc\delta n$, by this value of V, we get for the capacity of the ellipsoidal conductor,

$$C = \frac{2}{\displaystyle\int_o^\infty \frac{d\lambda}{(a^2+\lambda)^{\frac{1}{2}}(b^2+\lambda)^{\frac{1}{2}}(c^2+\lambda)^{\frac{1}{2}}}} \quad . \quad . \quad . \quad (51)$$

Ellipsoid of Revolution.

The integral can be found in finite terms if the ellipsoid be one of revolution. Putting for this case $a = b$, and transforming the integral by writing $1/y^2$ for $a^2 + \lambda$, we easily get

$$\int_o^\infty \frac{d\lambda}{(a^2+\lambda)(c^2+\lambda)^{\frac{1}{2}}} = \frac{2}{\sqrt{c^2-a^2}}\left\{\log\frac{\sqrt{(c^2-a^2)(a^2+\lambda)}}{\sqrt{c^2-a^2}+\sqrt{c^2+\lambda}}\right\}_o^\infty,$$

or

$$-\frac{2}{\sqrt{a^2-c^2}}\left\{\sin^{-1}\frac{\sqrt{a^2-c^2}}{\sqrt{a^2+\lambda}}\right\}_o^\infty,$$

Prolate Ellipsoid.

according as $c >$ or $< a$. Evaluating these expressions we find for a prolate ellipsoid

$$C = \frac{\sqrt{c^2-a^2}}{\log\dfrac{\sqrt{c^2-a^2}+c}{a}} = \frac{2ce}{\log\dfrac{1+e}{1-e}} \quad . \quad . \quad . \quad (52)$$

Oblate Ellipsoid.

and for an oblate ellipsoid

$$C = \frac{\sqrt{a^2-c^2}}{\sin^{-1}\dfrac{\sqrt{a^2-c^2}}{a}} = \frac{ae}{\sin^{-1}e} \quad . \quad . \quad . \quad . \quad (53)$$

where e is the eccentricity in each case.

Evaluation of the vanishing fractions which these values of C become when $c = a$, gives in each case for a spherical surface $C = a$, the result otherwise obtained above (p. 47).

If now c be so great in comparison with a that a^2/c^2 may be neglected, (52) becomes Capacity of a thin Cylinder.

$$C = \frac{c}{\log \frac{2c}{a}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

the capacity of a right circular cylinder whose length $2c$ is great in comparison with its diameter $2a$.

If c be so small in comparison with a that c^2/a^2 may be neglected (53) becomes Capacity of a thin Circular Disc.

$$C = \frac{2a}{\pi} = \frac{a}{1\cdot 5708\ldots} \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

the capacity of a thin circular disc of radius a.

3. A Conducting Sphere surrounded by a concentric spherical conducting shell. Capacity of a Spherical Condenser.

Let r be the radius of the sphere, r_1 the internal radius and r_2 the external radius of the spherical shell, q the charge of the internal sphere, q' the independent charge of the shell. The potential of the inner sphere due to its own charge is q/r, the potential at every point within the outer sphere due to q' is q'/r_2. But the charge on the internal sphere produces an induced charge of amount $-q$, on the inner surface of the shell, and a charge $+q$ on the outer surface. The potential of the sphere is therefore

$$V = \frac{q}{r} - \frac{q}{r_1} + \frac{q}{r_2} + \frac{q'}{r_2} = q\left(\frac{1}{r} - \frac{1}{r_1}\right) + (q + q')\frac{1}{r_2}.$$

When the shell is at zero potential, $(q + q')/r_2$ (the potential at its outer surface and therefore at every

Capacity of a Spherical Condenser.

point of it) is equal to zero. Hence we get for the capacity

$$C = \frac{r\,r_1}{r_1 - r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

The nearer therefore r and r_1 are made to equality, that is the smaller the distance between the inner and outer conductors, the greater is the capacity of the sphere.

Putting $r_1 - r = \tau$, and S for the surface of the internal conductor, we get instead of (56)

$$C = \frac{S}{4\,\pi\tau} + r \quad . \quad . \quad . \quad . \quad . \quad (57)$$

The external conductor therefore causes an addition of $S/4\pi\tau$ to the capacity of the sphere. If τ be very small this part of the capacity is very large in comparison with the other part r, the capacity of the sphere when alone, and we may put in this case

$$C = \frac{S}{4\pi\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Effect of Intermediate Conducting Shells.

If several conducting shells each without charge be placed between the outer and inner conductors, and the outer conductor be kept at zero potential; then if $\tau_1, \tau_2, \tau_3, \ldots \tau_{n-1}$ be the thicknesses, and $r_1, r_2, r_3, \ldots r_{n-1}$, the internal radii of these shells, and r, r_n the radii of the inner and outer conductors, the expression for the potential of the sphere becomes

$$V = q\left(\frac{1}{r} - \frac{1}{r_n} - \frac{1}{r_1} + \frac{1}{r_1 + \tau_1} - \&c. + \frac{1}{r_{n-1} + \tau_{n-1}}\right)$$

$$= q\left(\frac{1}{r} - \frac{1}{r_n} - \frac{\tau_1}{r_1(r_1 + \tau_1)} - \&c. - \frac{\tau_{n-1}}{r_{n-1}(r_{n-1} + \tau_{n-1})}\right)$$

If the thickness of each shell is small in comparison with either radius of the shell this becomes

Effect of Intermediate Conducting Shells.

$$V = q\left(\frac{1}{r} - \frac{1}{r_n} - \frac{\tau_1}{r_1^2} - \frac{\tau_2}{r_2^2} - \&c. - \frac{\tau_{n-1}}{r^2_{n-1}}\right)$$

For the capacity * of the inner conductor we have

$$C = \frac{rr_n}{r_n - r - \left(\frac{\tau_1}{r_1^2} + \frac{\tau_2}{r_2^2} + \&c. + \frac{\tau_{n-1}}{r^2_{n-1}}\right)rr_n}$$

Hence if $r_n - r$ be small in comparison with r and r_n we get

$$C = \frac{S}{4\pi\{r_n - r - (\tau_1 + \tau_2 + \ldots + \tau_{n-1})\}} \tag{59}$$

The effect of the intermediate shells is therefore simply to virtually diminish by their united thickness the distance between the inner and outer conductors.

4. A Conducting Cylinder of circular section enclosed within a coaxial conducting shell.

Long Cylindric Condenser or Submarine Cable.

We shall suppose the length $2c$ of the cylinder to be great in comparison with the respective diameters $2a$, $2b$ of the cylinder and the internal surface of the shell, and consider only parts of the inner and outer cylinders at distances from the ends great in comparison with either diameter. Such parts of the inner cylinder may be regarded as within an infinitely long cylindric shell, that is the distribution on both cylinders may be taken as uniform and the effects of the ends neglected.

* Since the intermediate shells are not at zero potential, the word "capacity" is here used in a somewhat different sense from that assigned to it in the definition (p. 46). It here means simply *charge per unit of potential* of the inner cylinder.

Long Cylindric Condenser or Submarine Cable.

By equation (54) the capacity of the inner cylinder per unit of length would, if there were no external shell, be $1/\left(2 \log \frac{2c}{a}\right)$; and therefore if σ be the surface density upon it, its potential would be $4\pi\sigma a \log (2c/a)$. But if the exterior shell be at potential zero there must be on its inner surface a distribution equal to that on the interior cylinder but opposite in sign. The potential within the shell produced by this distribution is $-4\pi\sigma a \log (2c/b)$, its value at the inner surface of the shell. Hence the total potential at the interior cylinder is given by the equation

$$V = 4\pi\sigma a \log \frac{b}{a},$$

and if C be the capacity of the cylinder per unit of length

$$C = \frac{1}{2} \cdot \frac{1}{\log \frac{b}{a}} \quad . \quad . \quad . \quad . \quad . \quad (60)$$

The same result may also be found as follows, by considering both cylinders as infinite in length, and integrating Laplace's equation for the space between them. Taking the origin on, and the axis of x along the axis of the cylinder, we have $d^2V/dx^2 = 0$, and Laplace's equation in the form

$$\frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0.$$

Putting $y = r\cos\theta$, $z = r\sin\theta$, we transform this to

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0.$$

By two successive integrations this gives

$$\frac{dV}{dr} = A\frac{1}{r}; \quad V = B + A\log r.$$

Long Cylindric Condenser or Submarine Cable.

Putting $\frac{dV}{dr} = -4\pi\sigma$ for $r = a$, and $V = 0$ for $r = b$, in these results, we get $A = -4\pi\sigma a$, $B = 4\pi\sigma a \log b$. Hence the potential of the inner cylinder is

$$V = 4\pi\sigma a \log \frac{b}{a},$$

and the capacity per unit of length

$$C = \frac{1}{2} \cdot \frac{1}{\log \frac{b}{a}}.$$

Case of two Parallel Plates.

5. Two parallel conducting plates.

This case is important in its application to the construction of electrometers and of standard condensers. Let AB, CD represent two parallel plates at a distance d

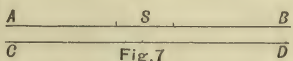

Fig.7

apart, small in comparison with any dimension in the plane of either plate. Let the potential of AB be V, of CD zero. Consider the charge on a portion of AB of area S, every point of which is at a distance great compared with d from any edge of either plate. The field of force at S between the plates must be uniform, and of intensity V/d. We have therefore, by Coulomb's law, if the electric density on the disc be σ, $\sigma = V/4\pi d$, and for the whole charge Q on S, $Q = VS/4\pi d$. The capacity C of the disc is therefore given by the equation

$$C = \frac{S}{4\pi d} \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

Case of two Parallel Plates.

For the energy E of the charge on S we have

$$E = \tfrac{1}{2} VQ = \frac{V^2 S}{8\pi d} = \tfrac{1}{2} V^2 C.$$

Let the total force on S towards the opposite plate be F, then the density of the distribution on the plate CD must, except near the edges, be $-\sigma$, and the attraction towards CD of a unit of electricity at any point on the disc has the value $2\pi\sigma$.* Therefore

$$F = 2\pi\sigma \times \sigma S = \frac{V^2 S}{8\pi d^2} \quad . \quad . \quad . \quad . \quad (62)$$

Hence

$$V = d\sqrt{\frac{8\pi F}{S}} \quad . \quad . \quad . \quad . \quad . \quad (63)$$

Guard-ring Condenser.

This equation is of use in the theory of the attracted disc electrometer (Chapter IV.). In such electrometers and in standard air condensers, made with parallel and movable plates, a portion at the centre of one of the plates is everywhere separated from the surrounding part, which is in the same plane, by a narrow gap.† The surrounding part of the plate has been called

* Equation (23), p. 15, gives the force on a unit of positive electricity at a point P on the axis of a circular disc of uniform density σ. When h is small compared with r the expression becomes $2\pi\sigma$. Since circular discs of different radii fulfilling this condition all give $2\pi\sigma$, the normal forces due to the parts of the disc at distances from P greater than the distance, r, of the nearest part of the edge, may be neglected, and the normal force at P is $2\pi\sigma$, whatever the form of the disc may be.

† For a full description of the arrangement in different cases see Chap. IV.

by Sir W. Thomson (to whom the arrangement is due) the guard-ring, and the inner portion the attracted disc. Supposing the attracted disc and guard-ring connected, they may be regarded as forming an arrangement deviating electrically only very slightly * from a continuous plane plate. By connecting the disc to the guard-ring, charging the guard-ring and disc as described above to potential V, and then breaking the connection without producing discharge, a charge $Q = VS/4\pi d$ is left on the disc. If then the force F be measured, we have

Guard-ring Condenser.

$$Q = \sqrt{\frac{FS}{2\pi}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

The following arrangement of conductors is important for its applications, especially to symmetrical electrometers. It consists of three conductors maintained at different potentials, and fulfilling the following conditions :—One of the conductors (A) (in the quadrant electrometer, Chapter IV., the needle) is symmetrically

Fig.8

placed with reference to the other two (B and C), and is so formed that one of its two ends or bounding edges is well under cover of B, and the other end or edge under cover of C, so that the electric distribution near each

Theory of Symmetrical Electrometer.

* For the amount of the deviation see Maxwell, *Elect. and Mag.*, vol. i. pp. 284, 307 (sec. ed.).

Theory of Symmetrical Electrometer.

end or edge is uninfluenced except by the nearer conductor. One such simple symmetrical arrangement is shown in the figure. Let the potentials of A, B, C be respectively V, V_1, V_2; and let A be slightly displaced from B towards C. This displacement may be angular or linear, according to the arrangement adopted; in the quadrant electrometer it is measured by the angle through which the needle is turned. Let θ denote the displacement and k the electrostatic capacity of A per unit of θ at places not near the ends or bounding edge of A, and well under cover of B and C. Then the quantity of electricity lost by A in consequence of its displacement relatively to B is $k\theta\,(V - V_1)$, and the quantity lost by B is $k\theta\,(V_1 - V)$. Similarly the quantities gained by A and C in consequence of the motion of A towards C are respectively $k\theta\,(V - V_2)$ and $k\theta\,(V_2 - V)$. Multiplying the first and second of these quantities by V and V_1 respectively, the third and fourth similarly by V and V_2, subtracting the sum of the first two products from the sum of the second two, and dividing by 2, we get for the work done by electrical forces in the displacement the value

$$k\theta\,(V_1 - V_2).\left(V - \frac{V_1 + V_2}{2}\right).$$

But this must be equal to the average couple multiplied into the displacement if the latter is angular, or the average force into the displacement if the latter is linear. We have therefore, denoting the force or couple by F,

$$F = k\,(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right) \quad . \quad . \quad (65)$$

In an arrangement of this kind when the displacement is small the couple or force acting on A is nearly the same over the whole displacement, and thus is nearly equal to the equilibrating couple or force due to the torsion wire, or bifilar, or other arrangement producing equilibrium. But for small displacements this will generally be proportional to the displacement, and therefore also to the deflection D on the scale of the instrument, and thus

Theory of Symmetrica Electrometer.

$$D = m\theta = c\,(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right) \quad . \quad (66)$$

where m and c are constants.

When V is great in comparison with V_1 and V_2 this reduces to $\theta = c'\,(V_1 - V_2)$ the equation employed when, as in the ordinary use of the quadrant electrometer, the needle is kept charged to a constant high potential.

Cylindric Sliding Condenser.

Let the conductors be cylinders, A of radius a, B and C of radius b, and let A be connected to B so that $V = V_1$, while C is maintained at potential zero. Also let A be mounted so as to be movable through measured distances in the direction of the axis. Since $V = V_1$, and $V_2 = 0$, a displacement of A through a distance x to the right or left will (60) respectively increase or diminish the capacity of A by an amount $x \Big/ 2 \log \dfrac{b}{a}$.

The arrangement thus made constitutes a condenser, the capacity of which can, when a and b are known, be altered through a considerable range of accurately determinate values. The construction and use of the instrument, which is due to Sir William Thomson, is described in Chapter IV. below.

Section IV.

GREEN'S THEOREM. INVERSE PROBLEMS. ELECTRIC IMAGES.

Proof of Green's Theorem.

We shall now prove Green's celebrated theorem to which we shall have to refer from time to time in what follows. Let U, V, be two finite, continuous and single-valued functions of x, y, z, the coordinates of a point within a closed surface S_0 (Fig. 9), and a a constant or any given function of x, y, z; and let also $\frac{dU}{dx}$, &c., $\frac{dV}{dx}$,

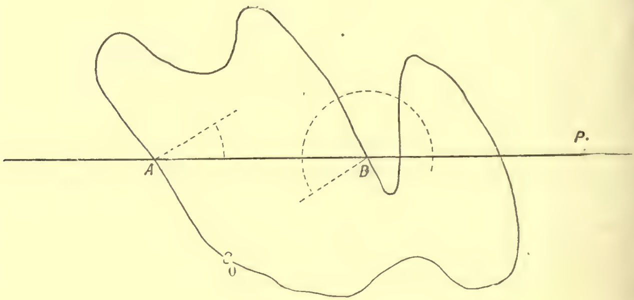

Fig. 9.

&c., be finite and continuous functions of x, y, z. Denoting by E the integral

$$\iiint a^2 \left(\frac{dU}{dx}\frac{dV}{dx} + \frac{dU}{dy}\frac{dV}{dy} + \frac{dU}{dz}\frac{dV}{dz}\right) dx\, dy\, dz$$

taken throughout the closed surface S, and integrating by parts, we get

$$E = \iint Ua^2 \left(\frac{dV}{dx}\, dy\, dz + \frac{dV}{dy}\, dz\, dx + \frac{dV}{dz}\, dx\, dy\right)$$

$$-\iiint U \left\{\frac{d}{dx}\left(a^2\frac{dV}{dx}\right) + \frac{d}{dy}\left(a^2\frac{dV}{dy}\right) + \frac{d}{dz}\left(a^2\frac{dV}{dz}\right)\right\} dx\, dy\, dz; \quad (67)$$

and a similar expression for E, in which U and V are interchanged

Proof of Green's Theorem.

Here the triple integral is taken throughout the space within the surface; and the terms of the double integral are taken as negative where a point moving in the positive direction along x, y, or z, as the case may be, enters the surface, and as positive where the point emerges. Considering the motion of the point parallel to the axis of x, let a normal be drawn *inwards* to the surface at each of the points of entrance and of emergence. If l_1, l_2 be the cosines of the angle which the normal makes as at A, B, with the positive direction of the axis of x at an entrance and an emergence respectively, and if ds_1, ds_2 be elements there of the surface, taken with their positive sides turned inwards, we have $dydz = l_1 ds_1$ at an entrance, and $dydz = -l_2 ds_2$, at an emergence. Hence for each pair of elements the corresponding part of the integral is

$$-\left(Ua^2 l\,\frac{dV}{dx}\,ds\right)_1 - \left(Ua^2 l\,\frac{dV}{dx}\,ds\right)_2,$$

and since we can exhaust the whole surface by pairs of elements we have for the first term of the integral

$$-\iint Ua^2 l\,\frac{dV}{dx}\,ds$$

taken over the surface. Putting m, n for the cosines of the angles between the normal and y, z, at any point, and proceeding in the same manner we get for the whole surface integral

$$-\iint Ua^2\left(l\,\frac{dV}{dx} + m\,\frac{dV}{dy} + n\,\frac{dV}{dz}\right)ds$$

Denoting the expression between the brackets, which is the rate of variation of V inwards along the normal, by $\frac{dV}{d\nu}$, and using $\frac{dU}{d\nu}$ in the same sense for U, we have finally

$$E = -\iint Ua^2\,\frac{dV}{d\nu}\,ds - \iiint U\left\{\frac{d}{dx}\left(a^2\,\frac{dV}{dx}\right)\right.$$
$$\left. + \frac{d}{dy}\left(a^2\,\frac{dV}{dy}\right) + \frac{d}{dz}\left(a^2\,\frac{dV}{dz}\right)\right\} dx\,dy\,dz$$

Proof of Green's Theorem.

$$= -\iint Va^2 \frac{dU}{d\nu}\,ds - \iiint V\left\{\frac{d}{dx}\left(a^2\frac{dU}{dx}\right) + \frac{d}{dy}\left(a^2\frac{dU}{dy}\right) + \frac{d}{dz}\left(a^2\frac{dU}{dz}\right)\right\} dx\,dy\,dz \quad (68)$$

which is Green's Theorem.

Discontinuity of $\frac{dU}{dx}$ &c.

If the space rate of variation of one of the functions in any direction, say $\frac{dU}{dx}$, is discontinuous within the limits of integration, the term $\frac{d}{dx}\left(a^2\frac{dU}{dx}\right)$ in the second expression for E becomes infinite, and the triple integral involving this term cannot be evaluated. Let P (Fig. 10) be a point within the

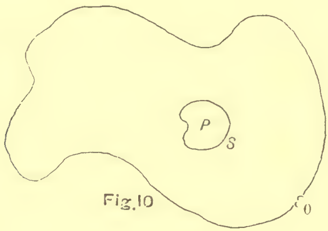

Fig. 10

space considered, at which $\frac{dU}{dx}$, &c., are discontinuous, and let a small closed surface S including P be described, then (since evidently, the theorem applies to *any* portion of space provided the surface integral is taken over all the bounding surface or surfaces), we can find the value of E by taking the triple integral of the second expression through the rest of the space, and adding to the surface integral the value of $-\iint Va^2\frac{dU}{d\nu}\,ds$ taken over S. By making S small enough we can obtain as nearly as we please the true and finite value of E.

Now suppose that the space variation in any direction is discontinuous at a surface If this surface be open we may imagine a closed surface described enclosing it, and then made to contract until it forms an infinitely thin shell, S (represented by the dotted line, Fig. 11), with the surface of discontinuity everywhere between

its faces. We find E, then, for the rest of the space within the external containing surface S_0 by adding to the surface integral over S_0 the value of $-\iint Va^2 \frac{dU}{d\nu} ds$ taken over the internal surface S, the normal being drawn as before at each point from the surface into the space through which the integral is taken.

Discontinuity of $\frac{dU}{dx}$ &c.

FIG. 11.

If the surface of discontinuity be closed, we have only to suppose a surface S (Fig. 12) described around it everywhere infinitely near it; and adding to the surface integral over S_0 $-\iint Va^2 \frac{dU}{d\nu} ds$

Fig. 12

taken over S, and taking the triple integral through the space between S and S_0, we see that the theorem holds for this space. The theorem holds also for the space within the surface of discontinuity, when the external boundary is taken everywhere infinitely near that surface on the inner side. The theorem thus

Discontinuity of $\frac{dU}{dx}$ &c.

holds separately for these two spaces, and therefore for both together, when the surface integral over both sides of the surface of discontinuity is taken into account.

In the same way we may deal with any number of surfaces of discontinuity arranged in any manner, and we see that it is only necessary to integrate throughout the spaces in which the space variations of U are continuous, and add the values of $-\iint Va^2 \frac{dU}{d\nu}ds$ for both sides of each surface of discontinuity.

Further, it is important to notice that if any portions of space are separated from the rest of space and from one another by closed or infinite surfaces, we may treat them as independent portions of space, and apply the theorem to them separately, being careful to include the surface integral over each bounding surface.

Surface Integral of Electric Induction derived from Green's Theorem.

We shall now give some applications of Green's Theorem. First let $U = 1$, so that we have $\frac{dU}{dx} = 0$, &c., also let $a = 1$, and V be the potential of any distribution of electricity of volume density ρ at any point within the closed surface S_0. Applying the theorem to the space within S_0, we get $E = 0$, and therefore the remarkable relation

$$-\iint \frac{dV}{d\nu}\,ds = \iiint \nabla^2 V\,dx\,dy\,dz. \quad . \quad . \quad . \quad (69)$$

Since $\frac{dV}{d\nu}$ is the space rate of variation inwards along the normal it is equal to the normal force N outwards; and, since (p. 10) $\nabla^2 V = 0$ where there is no electricity, and [as can be proved independently of (13)] $= -4\pi\rho$ where the electric density is ρ,

$$\iint N ds = 4\pi \iiint \rho\,dx\,dy\,dz.$$

which agrees with the theorem of eq. (13).

Again, if the portion of the distribution within the surface have nowhere finite density ρ, but consist of a surface distribution, of density σ at any point, we can apply the theorem by taking account of the corresponding discontinuities in the values of dV/dx, &c. Let first the surface, S, on which the electricity is distributed be a single open surface, as in Fig. 11; V_1, V_2 denote the potentials at two infinitely near points on opposite sides of it, and ν_1, ν_2

normals at those points drawn from the surface into the space between it and S_0. We have putting ds_0 for an element of S_0

Surface Integral of Electric Induction derived from Green's Theorem.

$$-\iint \frac{dV}{d\nu}\, ds_0 - \iint \left(\frac{dV_1}{d\nu_1} + \frac{dV_2}{d\nu_2}\right) ds = 0 \quad . \quad . \quad (70)$$

where the two parts of the integral in the second term are over the opposite portions of a surface enclosing, and infinitely nearly coincident with S (ds being taken to denote an element of either), and therefore may be regarded as taken over S. But proceeding as at p. 36, taking ρ finite, we get for Poisson's equation $d^2V/d\nu^2 = -4\pi\rho$. Hence integrating over the thickness, τ, of the stratum, and putting $\rho_1\tau = \sigma$ when τ is diminished indefinitely and ρ_1, the average density, is correspondingly increased, we get

$$\frac{dV_1}{d\nu_1} + \frac{dV_2}{d\nu_2} + 4\pi\sigma = 0.$$

Hence (70) becomes

$$\iint N ds_0 = 4\pi \iint \sigma ds,$$

where N is the normal force at ds_0 in the *outward* direction from S_0. This also agrees with the theorem of eq. (13).

The same result applies to a closed surface (which is a particular case of an open surface, with opening infinitely small), in which case with the condition of zero density in the interior, $dV/d\nu = 0$, on the inner side; and so for any group of surfaces, closed or unclosed, on which there is electricity with finite surface density—that is, at which the electric force changes abruptly.

Energy Equation proved analytically.

Putting $U = V$, and $a = 1$, we get

$$\iiint \left\{\left(\frac{dV}{dx}\right)^2 + \left(\frac{dV}{dy}\right)^2 + \left(\frac{dV}{dz}\right)^2\right\} dx\, dy\, dz$$
$$= -\iint V \frac{dV}{d\nu}\, ds - \iiint V . \nabla^2 V\, dx\, dy\, dz. \quad (71)$$

Let V denote the potential at any point of any finite distribution of electricity, and let the triple integral on the left be taken through all space. The surface integral on the right includes the surface integrals, which, as we have shown, belong to each surface distribution. Wherever there is a volume distribution we have $\nabla^2 V = -4\pi\rho$, elsewhere $\nabla^2 V = 0$. Again $-\iint V \dfrac{dV}{d\nu}\, ds$ taken over the external closed surface S_0 becomes zero when S_0 is at

Energy Equation proved analytically.

an infinite distance from the electrical distribution, for $\iint \frac{dV}{dv}\, ds$ is constant for all distances at which S_0 encloses the whole distribution, and $V = 0$.

The part of the surface integral depending on the electrified surfaces is $-\iint \left(\frac{dV_1}{dv_1} + \frac{dV_2}{dv_2}\right) ds$ taken over all the surfaces, and this as we have seen is $4\pi \iint \sigma ds$ taken over the same surfaces. Hence we find, putting F^2 for the quantity in brackets on the left, and dividing by 8π,

$$\frac{1}{8\pi} \iiint F^2\, dx\, dy\, dz = \tfrac{1}{2} \iint V\sigma\, ds + \tfrac{1}{2} \iiint V\rho\, dx\, dy\, dz,$$

the energy equation found synthetically above.

Solution of Laplace's Equation for given surface conditions is unique.

It has been shown analytically (Thomson and Tait's *Nat. Phil.* vol. i. part i. App. A. (d),) that a function V exists which has a given value for each point of any surface or surfaces in the electric field, and satisfies the equation $\nabla^2 V = 0$ at every other point. This is the case of an electrified system bounded by surfaces at which the potential is given in a dielectric containing no electricity external to these surfaces, and since the conditions of the problem are physically possible it must have at least one solution. A solution of this equation with the given surface condition therefore exists, and we can prove easily that there is only one such solution.

From this it follows that if the potential be given over any surfaces in the electric field it is determinate throughout the rest of the field, in the presence there of any given electric distribution. For the potential at each point due to the given distribution is everywhere determinate (provided the electric surface or volume density is finite at every point); hence if V_1 denote the potential at any point of the surface due to the given distribution, and V be the actual potential at the same point, then $V - V_1$ can be found for each point of the surface, and this will be the system of surface values of the function which satisfies $\nabla^2 V = 0$, of which there exists a solution.

We shall now prove that

1. If the potential be given at each point of a surface or system of surfaces described in an electric field, then for any point of the field for which there exists a finite value of the potential there is only one such value. The surfaces may be open or closed, and there may be any given electric distribution in the space, whether within or without the given surfaces.

For let a finite value of the potential at any point P be V, then V must satisfy the characteristic equation

Solution of Laplace's Equation for given surface conditions is unique.

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + 4\pi\rho = 0,$$

where ρ is the (finite or zero) value of the volume density of the electric distribution at P; or if an electrified surface exists in the field, the values of the rates of variation of V *from* the surface along normals ν, ν' drawn at any point must satisfy the equation

$$\frac{dV}{d\nu} + \frac{dV'}{d\nu'} + 4\pi\sigma = 0.$$

Let, if possible, V_1 be another value of the potential at P, equal to V, the given value at each point of the given surfaces and satisfying these equations everywhere else. We see at once that a potential $V - V_1$, satisfies all the conditions for the case in which the potential is zero at the given surfaces, and both σ and ρ are zero everywhere else. But in this case, since the potential is also zero at an infinite distance, it must be zero everywhere else, otherwise there would be one or more points of maximum or minimum potential in space void of electricity. Hence $V = V_1$, that is, there is only one value of V which satisfies the equations at every point, and coincides with the given value for every point on the surface.

2. There is one and only one distribution of electricity over a given surface or system of surfaces in the electric field which for a given distribution elsewhere than on the surface corresponds to an arbitrarily given potential at each point of the surface; and only one value of the potential can be produced at each point of the surface when a given distribution of electricity is made over the surface.

Surface Distribution producing given Potential at every point, and Potential produced by any given Distribution, both unique.

If V, V', the potentials on opposite sides of the surface, be arbitrarily given at every point we have seen that the potential is single valued at every point in the field. Hence the density $-\frac{1}{4\pi}\left(\frac{dV}{d\nu} + \frac{dV'}{d\nu'}\right)$ which, p. 28 above, must exist on the surface is also single valued.

Again, if this distribution be made, the given potentials at the surface will be produced. For, if not, let V_1, V_1', instead of V, V', be the potentials, on the two sides of the surface, produced at any point.

Surface Distribution producing given Potential at every point, and Potential produced by any given Distribution, both unique.

We have then the two equations

$$\frac{dV_1}{d\nu} + \frac{dV_1'}{d\nu'} + 4\pi\sigma = 0.$$

$$\frac{dV}{d\nu} + \frac{dV'}{d\nu'} + 4\pi\sigma = 0.$$

Subtracting we get

$$\frac{d(V - V_1)}{d\nu} + \frac{d(V' - V_1')}{d\nu'} = 0.$$

Hence $V - V_1$, $V' - V_1'$ are values of the potential at points infinitely near one another on opposite sides of the surface in the case in which σ is zero, and, since Poisson's equation is by hypothesis satisfied in the case both of V, V', and V_1, V_1', we see that $V - V_1$, $V' - V_1'$ correspond also to the case in which ρ is zero. But when σ and ρ are everywhere zero the potential is everywhere zero, and we have $V = V_1$, $V' = V_1'$; that is, the distribution does produce the given potential.

Green's Problem.

From (2) we see that electricity can be distributed in one, and only one, way on any surface or surfaces in the electric field so as to produce, with any other given distribution in the field, any required potential at any point infinitely near the surfaces, and that the potential at any other point is perfectly determinate. The density of the distribution at each point must be $-\frac{1}{4\pi}\left(\frac{dV_2}{d\nu_1} + \frac{dV_2}{d\nu_2}\right)$, and therefore there is only one quantity of electricity which can be thus distributed.

Solution.

It was proved by Green that if σ be the surface density required at any element E (Fig. 13) of a surface in order to produce by its

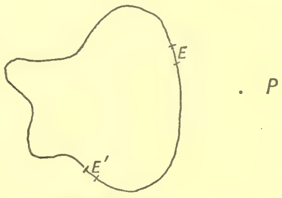

FIG. 13.

own action a potential infinitely near that point, equal to that produced by a unit of electricity at a point P not on the surface,

the potential V at P due to a surface distribution which produces any arbitrary potential $f(E)$ at E is given by the equation Solution.

$$V = \iint \sigma f(E) ds \quad . \quad . \quad . \quad . \quad . \quad . \quad (72)$$

the integral being taken over the surface. For if E' be any other element and σ' the density there, corresponding to σ at E, we have

$$\iint \frac{\sigma' ds'}{EE'} = \frac{1}{EP}.$$

If σ_1 be the density at E of the distribution required to produce potential $f(E)$ at E we have for the potential at P produced by this distribution

$$V = \iint \frac{\sigma_1 ds}{EP} = \iint \sigma_1 ds \iint \frac{\sigma' ds'}{EE'},$$

by the last equation. But this integral may be written

$$\iint \sigma' ds' \iint \frac{\sigma ds}{EE'},$$

and by the definition of σ_1

$$f(E) = \iint \frac{\sigma_1 ds}{EE'}.$$

Hence

$$V = \int \sigma f(E) ds.$$

The value of σ is found below (p. 79) for the case of a spherical surface.

The direct problem which presents itself in electrostatics is the determination, for a given system of conductors with given charges, of the potential at every point of the field, and the density of the distribution at every point of the conductors. Direct Problem of Electrostatics.

It is easy to show that if the charges are given, there is only one possible distribution on the conductors, and it follows from what has been proved above, that the

Potentials and Distribution unique for given Charges.

potential at every point of the field is also unique in value. To prove that the distribution is unique consider a system of conductors the charges of which are separately zero. The potential over each conductor must for equilibrium be constant, and must be zero. For if not zero, the potentials must either have all the same positive or negative value, or have different values. In the latter case the potential cannot, since there is no maximum or minimum of potential in the field, increase outwards from one part, and diminish outwards from another part of the surface of the conductor, whose potential is numerically greatest. Hence the electric induction across every element of the surface has the same sign, that is, every element is electrified in the same manner. But this is impossible, since the whole charge is zero. In the case of all the conductors at one potential we may apply the same test, with the same result, to any conductor. We see, therefore, that the surface-density on each must be everywhere zero.

Conductors neutral when Charges zero.

Potentials and Distribution proved unique for given Charges.

Let now V_1, V_2, &c., and V'_1, V'_2, &c., be two possible systems of potentials corresponding to the given charges, then $-V'_1$, $-V'_2$, &c., will be a possible system of potentials for equal and opposite charges, and $V_1 - V'_1$, $V_2 - V'_2$ &c., possible potentials when their charges are zero. But in this last case we have seen that the conductors are not electrified, and therefore $V_1 = V'_1$, $V_2 = V'_2$, &c.

The potentials at the conductors are thus unique in value, and we have seen (p. 69) that the potential at any point in the field is also unique in value. The outward

force R at any point of each conductor is thus fixed in value, and since $R/4\pi$ is the electric surface density at the point, the charges can be distributed in only one manner.

Inverse Problems.

The direct problem stated above has only been solved in certain cases, but the inverse problem of finding a system of conductors and charges which will produce a given possible system of potentials, can be solved with comparative ease, and the results applied to the solution of cases of the direct problem.[1] We shall now give some examples of this mode of proceeding.

Surface Distribution replacing given system.

If any surfaces whatever, open or closed, be described in the electric field, it has been proved (p. 69 above) that it is possible to find one, and only one, distribution of electricity over these surfaces which shall produce at each point of them, and at each point of space entirely separated from the electric distribution by those of the surfaces which are closed or infinite, the same potential as is produced in the actual case. For example, let S_1, S_2, S_3 (Figs. 14 and 15) be three surfaces drawn in the electric field of the distributions, it is possible (p. 69) to find one, and only one, distribution over S_1, S_2, S_3, which shall produce over each of the surfaces, and throughout H_1 and H_2, the same potential as is produced by M. The potential at any point of K depends on the given surfaces, S_1, S_2, S_3, and since the potential at every point of the surfaces is given, is, as we have seen, perfectly determinate.

Further, the part of this distribution over any closed

[1] See Maxwell's *Elementary Treatise on Electricity*, p. 72 *et seq.*

Surface Distribution replacing given system.

or infinite surfaces separating a region within which lies any part of the actual distribution from the rest of space

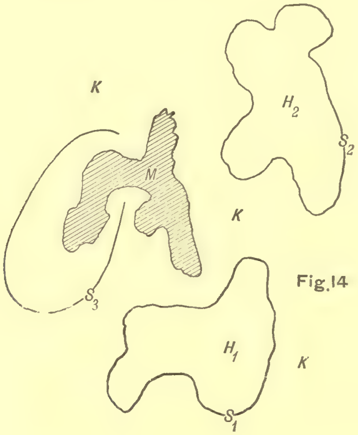

Fig. 14

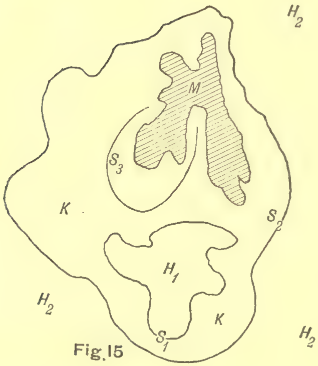

Fig. 15

is such as to exactly produce at each point of space external to that region the same potential as is pro-

Surface Distribution replacing given system.

duced by that part; for it is clear that the potential at any point of the surfaces will be that which would be produced by the actual distribution and (p. 68 above) the potential produced by the electrified surfaces at

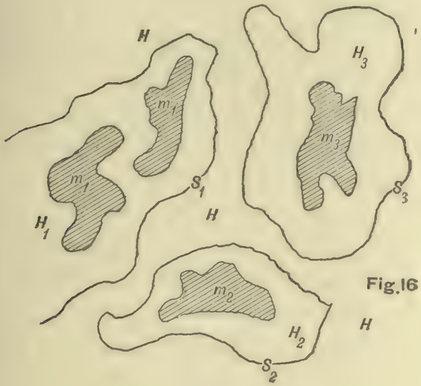

Fig. 16

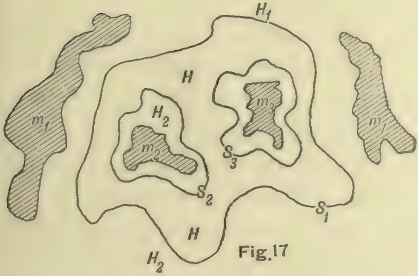

Fig. 17

every point of the space which they separate from the electric distribution coincides with that due to the actual distribution. For example, if H_1, H_2, H_3 (Figs. 16 and 17) be regions within which are distributions

m_1, m_2, m_3 separated by the surfaces S_1, S_2, S_3 from H, the rest of space, the distribution over S_1 produces the same potential throughout H_2, H_3, and H as is produced by m_1, and similarly for m_2, m_3. Hence the potential at each point of S_1, S_2, S_3, and throughout H, is that due to m_1, m_2, m_3 jointly.[1]

Distribution on Conducting Surface replacing given Internal system.

Suppose a surface S taken in the electric field to be an equipotential surface for the given distribution, we see from what immediately precedes that electricity can be distributed on that surface so as to produce potential in the space A, on one side of the surface, *equal to that produced by the electricity* (in some cases part only of the whole distribution, in others the whole) *in the space B, on the other side of the surface.* We may suppose the distribution on the surface made, and the original distribution in B removed. The potential throughout B must be constant, and the electric force there zero, and the resultant force at any point infinitely near the surface on the side A is normal to the surface. Hence if R be this normal force, taken positively when from the surface towards A, the electric surface density at the point is $R/4\pi$. This result had already been found (p. 29 above), but we have now seen that it is the unique solution of the problem. We shall find many examples of its utility in what follows.

Definition of an Electric Image.

The distribution in the space B which produces in the space A on the other side of the surface the same potential as is produced by the distribution supposed

[1] See Thomson and Tait, *Nat. Phil.* vol. i. part ii. p. 56, *et seq.*, from which Figs. 14—17 are taken, with slight modifications.

made as above on the surface is called an *Electric Image.*

Let now the system of conductors be three in number, a sphere S of radius a, and two external conductors, S_1, S_2, Fig. 18, of dimensions so small in comparison with a that a charge of electricity on either may be considered as concentrated at a point. Let the distance of S_1 from C, the centre of S, be f, the distance of S_2 from P r_{12}, and let S_1 be insulated and charged with a quantity q of electricity, S_2 insulated but uncharged, Sphere under Influence of Electric Charge at External Point.

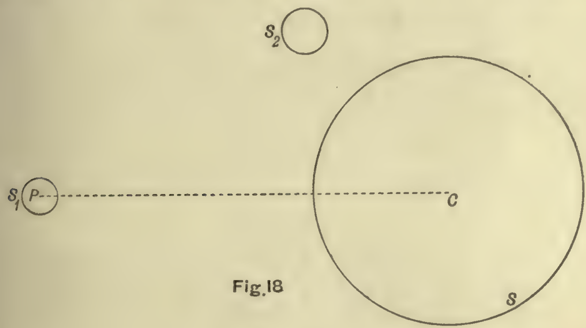

Fig. 18

and S uninsulated, and therefore at zero potential. By the theory given above (p. 38)

$$V_2 = p_{12}q + p_{s2}q_s$$

where p_{s2} is the mutual coefficient of potential of S_2 and S, p_{12} $(= 1/r_{12})$ that of S_1 and S_2, and q_s the induced charge on S. But since the potential of S is zero, and every element of the charge of S is at a distance a from the centre

$$\frac{q}{f} + \frac{q_s}{a} = 0 \text{ or } q_s = -\frac{a}{f}q. \quad . \quad . \quad . \quad (73)$$

Sphere under Influence of Electric Charge at External Point.

Taking S_2 infinitely small and infinitely near the surface of the sphere so that $V_2 = O$, we have for this case

$$p_{s2} = \frac{1}{r_{12}a/f} \quad . \quad . \quad . \quad . \quad . \quad . \quad (74)$$

Let E, Fig. 19, be the position of the centre of C_2, P' a point in PC taken so that $P'C : CE :: CE : CP$, that

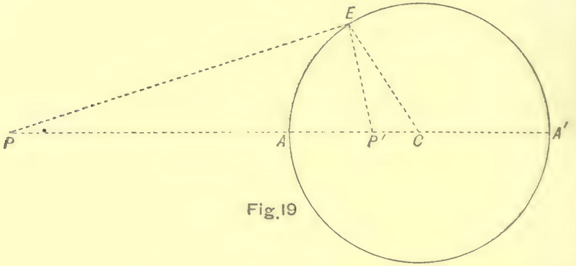

Fig. 19

is, so that the triangles PEC, $EP'C$, are similar, and $CP' = a^2/f$. Hence $EP' = r_{12}a/f$, and therefore

$$p_{s2} = \frac{1}{EP'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (75)$$

Electric Images in Spherical Surfaces.

Equation (73) shows that the amount of the induced charge on the sphere S, called forth by a charge q concentrated at P, is $- a/f \, . \, q$, and that the potential which this charge produces at each point of the surface of S, and therefore also at every external point, is the same as that which it would produce if concentrated at the internal point P', while the potential which it produces at every internal point is equal and opposite to that due to q at P.

Electric Image in Spherical Surface.

Passing from potentials to forces, the force at every external point due to the induced distribution is the same as that which would be produced by $-a/f \, . \, q$ at P', and at every internal point is equal and opposite to that produced by the charge of amount q at P.

We can now find the distribution on S. The resultant force on a unit of positive electricity at E is that due to an attraction $-\frac{a}{f}\frac{q}{P'E^2}$ along EP' and a repulsion $\frac{q}{PE^2}$ along PE. Resolving these forces along PC and EC we have for the component in the direction PC, the expression $q\frac{CP}{PE^3} - q\frac{CP'}{P'E^3}\frac{a}{f}$, and for the component in the direction EC, $-q\frac{CE}{PE^3} + q\frac{CE}{P'E^3}\frac{a}{f}$. Since $P'E = PE\frac{a}{f}$ and $CP' = a^2/f$ the former expression vanishes, and the radial component, which from the vanishing of the other component we see is the resultant force, is $-q\frac{1}{PE^3}\frac{f^2-a^2}{a}$. By Coulomb's theorem, if σ be the density at E, we get from this expression and Fig. 19

$$\sigma = -q\frac{PA \, . \, PA'}{4\pi \, . \, CE} \cdot \frac{1}{PE^3} \quad . \quad . \quad . \quad . \quad (76)$$

The density at any point of the surface is therefore inversely as the third power of the distance of the point from that at which the inducing charge is situated.

The point P' is called, from the optical analogy of virtual images formed by reflection, the *Electric Image* of the point P in the spherical surface S. The external potential and electric force produced by the induced

Electric Image in Spherical Surface.

electrification on the surface called forth by q at P are those due to a *virtual* charge of amount $-a/f.q$ at P', that is, are the same as would be produced by this charge at P' if the conductor S did not exist.

We see easily that P and P' are conjugate to one another—that is, while P' is the image of P for the sphere S influenced by a charge at P, P is the image of P' for the same sphere influenced by a charge at P'. For suppose a charge q to be placed at P' within the conductor S supposed at potential zero, the external potential is everywhere zero—that is, the induced electrification produces an external potential and electric force everywhere equal and opposite to that produced by q. In this case, since q is within S, q_s, the induced charge on S, is $-q$. Proceeding as before with this value of q_s, we find that the internal potential and electric force due to the induced electrification is that due to a virtual charge $-q$ at P, and that for the density at any point E in S we get as before

$$\sigma = -q\,\frac{a^2 - f^2}{4\pi a}\,\frac{1}{P'E^3} = -q\,\frac{P'A\,.\,P'A'}{4\pi\,.\,CE}\,\frac{1}{P'E^3} \quad (77)$$

The induced electrification produced by any given external or internal electric system can be found by determining the distribution due to each point of the system and superimposing the distributions. Hence an electric system on the other side of the spherical surface is determined which would produce on the same side as the given system the same potential at every point as is produced by the induced distribution. This system is made up of elements which are the

electric images of the elements of the inducing system, of which it is therefore said to be the electric image.

Electric Image in Spherical Surface.

Thus let the system be a series of charges q_1, q_2, &c., at external points P_1, P_2, &c., whose distances from the centre of the sphere are f_1, f_2, &c., the charge on the sphere is then $-a\,\Sigma\frac{q}{f}$; and the density at any point is

$$-\frac{1}{4\pi a}\Sigma(f^2-a^2)\frac{q}{PE^3} \quad . \quad . \quad . \quad (78)$$

the summation being taken for every point of the inducing system.

If the system be internal the charge on the sphere is $-\Sigma q$, and the density is

$$-\frac{1}{4\pi a}\Sigma(a^2-f'^2)\frac{q}{P'E^3} \quad . \quad . \quad . \quad (78\ bis)$$

where $f' = P'C$.

If the sphere be at zero potential under the influence of both an external and an internal system, the charge on the sphere will be the sum of the charges and the density the sum of the densities in these two cases taken separately.

Induced Distribution on Insulated Sphere.

If the sphere be insulated and at potential V, the distribution is obtained by superimposing on the induced distribution just found a uniform distribution of density $V/4\pi a$. Hence we have for the density in this case

$$\left.\begin{array}{l} \sigma = \dfrac{1}{4\pi a}\left\{V - \Sigma\,(f^2-a^2)\,\dfrac{q}{PE^3}\right\} \\ \text{or} \\ \sigma = \dfrac{1}{4\pi a}\left\{V - \Sigma\,(a^2-f^2)\,\dfrac{q}{P'E^3}\right\} \end{array}\right\} \quad . \quad (79)$$

Induced Distribution on Insulated Sphere.

according as the inducing distribution is external or internal. Equivalent expressions for the density in terms of the charge Q on the sphere are easily obtained by substituting from the equation

$$V = \frac{Q}{a} + \frac{q}{f}.$$

It is instructive and easy to calculate from these expressions for a single inducing point-charge the density of the distribution on the sphere at the points nearest to and farthest from P, to find when these are opposite in sign, and in this case to determine the position of the circular line of zero density on the sphere for given values of Q, q, a, f.

Electric Energy of Two Spherical Conductors one of which is small:

We may consider a single point-charge as on a sphere of centre P, and radius b so small that the distribution on the small sphere may, so far as its effect on the distribution of the large sphere is concerned, be taken either as uniform or as concentrated at the point P. When the radii of the spheres are comparable with one another and with the distance between their centres, the distributions mutually influence one another, and the effect of either can only be expressed by an infinite series of electric images. When b is very small compared with a, all these images may be neglected except the first image of the smaller sphere. We have then the respective values q, $q\left(\frac{1}{b} - \frac{a/f}{f - a^2/f}\right) + Va/f$, for the charge and potential of the smaller sphere, and $Q\ (= Va - q \, . \, a/f)$, V, for those of the larger sphere.

Hence (p. 32) the electric energy, E, of the system of two conductors is given by the equation

$$\left.\begin{aligned} E &= \tfrac{1}{2}q^2\left(\frac{1}{b} - \frac{a}{f^2 - a^2}\right) + \tfrac{1}{2}V^2 a \\ &= \tfrac{1}{2}\frac{Q^2}{a} + \frac{Qq}{f} + \tfrac{1}{2}q^2\left(\frac{1}{b} - \frac{a^3}{f^2(f^2 - a^2)}\right) \end{aligned}\right\} \quad (80)$$

Mutual Force between them,

The work done by electric forces in separating the spheres by a small distance df will alter E by an amount $dE/df \, . \, df$, that is $-\, dE/df$ is the mutual repulsive force between the spheres in the line of centres. But

$$-\frac{dE}{df} = \frac{Qq}{f^2} - q^2 a^3 \cdot \frac{2f^2 - a^2}{f^3(f^2 - a^2)} = \frac{a}{f^2} Vq - \frac{af}{(f^2 - a^2)^2} q^2 . \quad (81)$$

when Attractive and when Repulsive.

The force will therefore be an attraction, zero, or a repulsion according as either of the equivalent expressions on the right-hand side is negative, zero, or positive. Hence it is an attraction if V or Q is zero, that is, if the sphere is uninsulated, or if it has no charge on the whole. Also if f is but little greater than a, that is, if the smaller sphere be very near the surface of the larger, $f^2 - a^2$ is very small, and the force is an attraction. If Q have the same sign as q, and be greater than $q \, . \, a^3 f(2f^2 - a^2)/(f^2 - a^2)$, or if V be positive and greater than $q \, . \, f^3/(f^2 - a^2)$, the force is a repulsion. These results explain the apparent anomaly of the attraction of a small charged sphere by a similarly charged conductor, when the distance between them is small, and the repulsion between them when they are at greater distances.

Images in Infinite Plane or in Sphere of large radius.

If d be the shortest distance of any one of the points P from the surface of the sphere, we have $f = d + a$, and $f^2 - a^2 = d\,(d + 2a)$. Hence if a is very great in comparison with d, that is, if the surface be an infinite plane (Fig. 20), or if the point P be near the surface, we have when $V = 0$

$$\sigma = -\frac{1}{2\pi}\sum\frac{qd}{PE^3} \quad . \quad . \quad . \quad . \quad . \quad (82)$$

Electric Image in Plane coincident with Optical Image.

Considering again the sphere of finite radius a, we have for the shortest distance of the image of any point-charge P from the sphere, the value $a - a^2/(d + a)$, or $d/(1 + d/a)$. Hence if a is very great in comparison with d, this value becomes approximately d, that is the image is on the normal to the surface and at the same

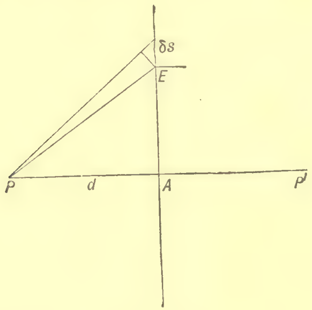

Fig. 20.

distance behind the surface that P is in front of it. Also since in this case $a/f = a/(a + d) = 1$ nearly, the image-charge is $-q$ to the same degree of approximation.

The distribution on an infinite plane might obviously

Direct Investigation for Infinite Plane.

have been obtained at once without considering the plane as a case of a sphere. For clearly the plane is a surface of zero potential for $+q$ at P and $-q$ at P', the optical image of P in the plane. The electrification of the plane is therefore equivalent, for all points to the left of the plane, to the charge $-q$ at P'. But the normal force outwards, that is, from the plane towards the space in which P is situated, is $-2qd/PE^3$, and the electric density at E is therefore $-qd/2\pi PE^3$. The results obtained above are thus verified.

It is easy to show that the integral of this expression for the density taken over the plane is $-q$, the value which it ought to have, since the space to the left of the plane may be supposed inclosed by the plane and a conductor at an infinite distance. Let δs be an element of the surface at E. The projection of δs on a plane at right angles to PE is $\delta s \,.\, d/PE$, and a cone with base δs, and vertex P will therefore intercept on a sphere of unit radius and centre P, an area $\delta s . d/PE^3$. The integral of this taken over the plane is therefore 2π, one-half the area of the unit sphere, and we get

$$-\frac{qd}{2\pi}\int\frac{\delta s}{PE^3} = -q.$$

Mutually Influencing Conductors.

When two or more conductors are electrified in the same field, the distribution on any one is influenced by that on the others, and hence the determination of the actual distribution becomes a problem of great difficulty, the solution of which has not been obtained except in certain cases. These have for the most part been solved only by the method of Electric Images combined

Mutually Influencing Conductors.

with the principle of Electric Inversion, by which the solution of any electric problem can be transformed into the solution of any number of other problems. These methods are due to Sir William Thomson, to whose papers we must refer for full information regarding them.[1] We give here, however, the solution of the problem of the distribution on two parallel conducting plane sheets of unlimited extent acted on by a point-charge placed between them, and a short explanation of the method of inversion with one or two examples of its use.

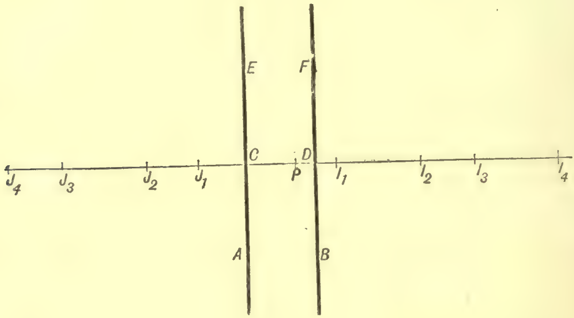

FIG. 21.

Two Parallel Planes with Point-Charge between them.

Let A, B (Fig. 21) be the traces of two parallel conducting planes of infinite extent maintained at zero potential, on a perpendicular plane through P, a point between them, at which a charge, q, is situated. Let a, β be the respective distances of P from C, D, the

[1] *Electrostatics and Magnetism*, pp. 52–97, and 146–191. See also Maxwell, *El. and Mag.* vol. i. sec. ed. pp. 226–260 *et seq.*

points at which a perpendicular through P meets the planes, and E a point on the plane A at a distance γ from C.

Induced Distribution replaced by Images.

If the plane B were removed the density at E on A due to a charge q at P would, (82) above, be $-q \cdot a/2\pi(a^2+\gamma^2)^{\frac{3}{2}}$. But the electrification of B, which would be produced by the charge q at P if A were removed, has at all points to the left of B the same effect as that of a quantity $-q$ at a point I_1 distant β from D to the right on PD produced, and the corresponding electric density at E is therefore

$$\frac{q}{2\pi}\frac{a+2\beta}{\{(a+2\beta)^2+\gamma^2\}^{\frac{3}{2}}}.$$

These two electrifications of A produce respectively the effects on the electrification of B of charges $-q$, $+q$, to the left of A on PC produced, the former at a point J_1 distant a, the latter at a point J_2 distant $a+2\beta$ from C. The electrifications of B thus produced have on A the effects of charges $+q$, $-q$ at points I_2, I_3, distant $3a+2\beta$, $3a+4\beta$, to the right of A on CD. The corresponding densities at E are therefore

$$-\frac{q}{2\pi}\frac{3a+2\beta}{\{(3a+2\beta)^2+\gamma^2\}^{\frac{3}{2}}},\quad \frac{q}{2\pi}\frac{3a+4\beta}{\{(3a+4\beta)^2+\gamma^2\}^{\frac{3}{2}}}.$$

In the same way another pair of densities at E could be found corresponding to point-charges $+q$, $-q$, at the respective distances $5a+4\beta$, $5a+6\beta$ to the right of A, and so on.

The electrification of A is that which would if B were removed be produced by $+q$ at P and an infinite

Induced Distribution replaced by Images.

trail of images I_1, I_2, I_3, &c., of charges $-q$, $+q$, $-q$, &c., at points to the right of P on CD produced. The potential at every point on A or to the left of it is plainly the potential due to $+q$ at P, and the image-charges to the right of P, and this is equal and opposite to the potential at the same point produced by the electrification of A. Similar results hold for B.

The potential at any point between the planes produced by the electrification of either is that due to the trail of images behind that plane, and the total actual potential at any such point is the sum of the potentials due to $+q$ at P and the two trails of images.

To verify that the potential of each of the planes is zero, let V be the potential at any point, E, of the plane A. Then we have

$$V = q\left(\frac{1}{PE} - \frac{1}{J_1E}\right) + q\Sigma\left(\frac{1}{I_{2n}E} - \frac{1}{I_{2n+1}E}\right) - q\Sigma\left(\frac{1}{J_{2n}E} - \frac{1}{J_{2n+1}E}\right),$$

where n has every integral value from 0 to ∞. Since $J_1E = PE$, the first term is zero; further each series is convergent, and the terms (which are arranged in the same order in both) are identically equal. Hence V is zero, and the above process of building up the electrification of each plane gives the required result.

Charges and Distances of Images.

The charges and distances of the images I_1, I_2, &c., are given by the table

Images	I_{2n-1},	I_{2n},
Charges	$-q$,	$+q$,
Distances from P	$2(n-1)(a+\beta)+2\beta$,	$2n(a+\beta)$.

where n has every positive integral value from 1 to ∞.

Charges and Distances of Images.

The charges and distances of the images J_1, J_2, &c., are given by the same table when a and β are interchanged.

Electric Density at any point.

By equation (82), p. 84, we have for the density at E

$$\sigma = \frac{q}{2\pi}\sum_{n=0}^{n=\infty}\left[-\frac{(2n+1)a+2n\beta}{\{[(2n+1)a+2n\beta]^2+\gamma^2\}^{\frac{3}{2}}}\right.$$

$$\left.+\frac{(2n+1)a+2(n+1)\beta}{\{[(2n+1)a+2(n+1)\beta]^2+\gamma^2\}^{\frac{3}{2}}}\right]. \quad . \quad (83)$$

where n is any positive integer.

The density at any point F on B (Fig. 21) is given by this equation when a and β are interchanged, and γ is taken as the distance from D to F.

Distribution on Two Planes cutting at right angles, represented by three images:—particular case of "Electric Kaleidoscope."

As an example of a case in which the number of images is finite, let the conductor consist of two planes at right angles to one another, terminated by their line of intersection, and influenced by a point-charge of amount $+q$ placed at a point P between them. Let O, OA, OB, Fig. 22, be the traces of the line of intersection and the planes on a plane drawn through P perpendicular to the conductor. Let a denote the distance of P from OA, β the distance of P from OB. The surfaces would evidently be at zero potential if insulated without charge and a charge $-q$ were situated at P_1 the image of P in OA, and at P_2 the image of P in OB, and a charge $+q$ at P_3, the common position of the image of P in OB and the image of P_2 in OA.

If N be the normal force at any point E of the plane

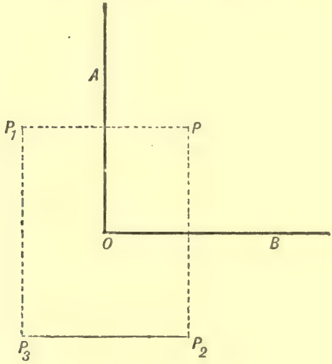

FIG. 22.

OA, taken positive towards the space BOA in which P is situated, we have

$$N = -2qa\left(\frac{1}{PE^3} - \frac{1}{P_2E^3}\right) \quad . \quad . \quad . \quad (84)$$

Electric Density at any point.

Since the space BOA may be regarded as the space external to a closed conductor, N is the actual force just outside the conductor at E, and we have for the electric density there

$$\sigma = -\frac{qa}{2\pi}\left(\frac{1}{PE^3} - \frac{1}{P_2E^3}\right) \quad . \quad . \quad . \quad (85)$$

Similarly for the density at any point F of OB we have

$$\sigma = -\frac{q\beta}{2\pi}\left(\frac{1}{PF^3} - \frac{1}{P_1F^3}\right) \quad . \quad . \quad (85\ bis)$$

Geometrical Inversion.

In Fig. 23 the distances CP, CP', CQ, CQ' fulfil the relation

$$CP \,.\, CP' = CQ \,.\, CQ' = a^2. \quad . \quad . \quad . \quad . \quad (86)$$

The point P' is called the inverse of the point P with respect to C, which is called the *centre of inversion;* and similarly Q' is the inverse of Q with reference to C. If

Geometrical Inversion

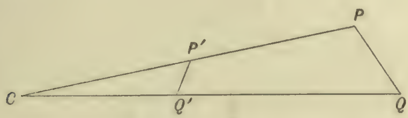

FIG. 23.

we have any system of points P, Q, &c., given, we can find in this manner a corresponding system of points P', Q', &c. If P, Q, &c., determine a surface or a solid figure, P', Q', &c., determine a corresponding surface or solid figure. The system P', Q', &c., is called the inverse of the system P, Q, &c., with reference to C; and it follows that the system P, Q, &c., is the inverse of P', Q', &c., with reference to the same point.

From what has gone before it is evident that any point P' is the image of the point P in the sphere of radius a and centre C. This sphere is called the *sphere of inversion* and its radius the *radius of inversion.*

Inverse of Point; its Image in Sphere of Inversion.

In Fig. 23 the triangles $CP'Q'$, CQP are similar. From this it follows that if P, Q be two very near points the angle CQP is equal to the angle $P'Q'Q$; that is, the lines QP, $Q'P'$ meet CQ at supplementary angles. Hence the angle which the inverse of any line or surface makes with any radius vector from the centre of inversion is the supplement of the angle which the original line or surface makes with the same radius vector. It follows that the inverses of any two lines or

Inversion—Geometrical Results.

Inversion —Geometrical Results.

surfaces intersect at the same angle as do the original lines or surfaces.

In particular it is easy to prove that the inverse of a circle is another circle, and therefore of a sphere another sphere. For let P, Q (Fig. 23) be the extremities of a diameter of the circle, and R (not shown in the figure) be any other point on the circle, then PRQ is a right angle. The inverse points are P', Q', R', and the angle $P'R'Q'$ is equal to a right angle $\pm$ the angle PCQ, according as CR does or does not intersect PQ. Hence, as R is moved round the circle, R' moves round another circle which is the inverse of the former.

Further the inverse of a straight line is a circle passing through the centre of inversion. For let P (Fig. 23) be a point on the straight line, such that CP is at right angles to PQ, and let Q be any other point on the line. Then if P', Q' be the corresponding inverse points, P' is fixed and $CQ'P'$ is a right angle for every position of Q'. Hence the locus of Q' is a circle of which the diameter is CP'.

Inversion —Geometrical Results.

The inverse of an infinite plane is obviously a spherical surface passing through the centre of inversion.

According as the centre of inversion is without or within the original surface, the space within the inverse is the inverse of the space within or without the original surface, and the space without the inverse is the inverse of the space without or within the original surface.

If $CQ = r$, $CQ' = r'$, and P, Q be points very near to one another, $PQ/P'Q' = r/r'$. Hence if r be the distance of any element δl of a line, δs of a surface, δv of a volume, and r' the distance of the corresponding inverse

elements $\delta l'$, ds', $\delta v'$, from the centre of inversion, we have

$$\delta l/\delta l' = r/r'\,; \quad \delta s/\delta s' = r^2/r'^2\,; \quad \delta v/\delta v' = r^3/r'^3. \quad (87)$$

Potential at an Inverse Point.

Again if we suppose at the point P (Fig. 23) a charge q, of which the potential at Q is $V (= q/PQ)$, then putting V or the potential at Q' of a charge aq/r (the corresponding image-charge with its sign changed) situated at the point P', we have $V' = aq/r\,.\,P'Q'$. Hence by the equations $CP\,.\,CP' = CQ\,.\,CQ' = a^2$,

$$V/V' = r'/a = a/r. \quad . \quad . \quad . \quad . \quad (88)$$

Inverse Electric Densities and Charges.

The quantity of electricity on an element δs of a surface is $\sigma\,.\,\delta s$, and on an element of volume $\rho\,.\,\delta v$. The corresponding image-charges with signs reversed are $\sigma\delta s\,.\,a/r$, $\rho\delta v\,.\,a/r$; and by (87) the corresponding areas and volumes are $\delta s' = \delta s\,.\,r'^2/r^2$, $\delta v' = \delta v\,.\,r'^3/r^3$. Hence if σ', ρ', q' be the image-densities and image-charge corresponding to σ, ρ, q respectively, we have

$$\sigma'/\sigma = a^3/r'^3 = r^3/a^3\,; \quad \rho'/\rho = a^5/r'^5 = r^5/a^5\,;$$
$$q'/q = r'/a = a/r \quad . \quad . \quad . \quad . \quad . \quad (89)$$

Electrical Inversion defined.

The process of inverting any electric system consists in inverting the geometrical arrangement of the system and substituting for every element of charge the corresponding element of image-charge with its sign changed.

If the potential of any distribution on a conductor in equilibrium be V at a point P, the potential of the inverse distribution at P' the image of P is by (88) Va/r', where

Equilibrium Distribution inverted

r' is the distance of P' from the centre of inversion C. Hence the potential at P' of the inverse distribution is the same as that of a quantity Va placed at C. The inverse distribution, and a quantity $-Va$ at C will keep P' at zero potential. If then we invert an equipotential surface of the given distribution, we shall obtain a surface of zero potential of the distribution composed of the inverse and $-Va$ at C. If we take as the equipotential surface the surface of a conductor on which the given distribution is in equilibrium, the inverse distribution is on a surface which, when $-Va$ is at C, is at zero potential. The inverse distribution is therefore the induced distribution on the inverse conductor under the influence of $-Va$ at C.

gives Induced Distribution due to Point-Charge at Centre of Inversion.

It follows that if we invert any distribution which gives at each point of itself distant r' from C a potential Va/r', that is, the induced distribution of a system $-Va$ at C, we get a system in equilibrium.

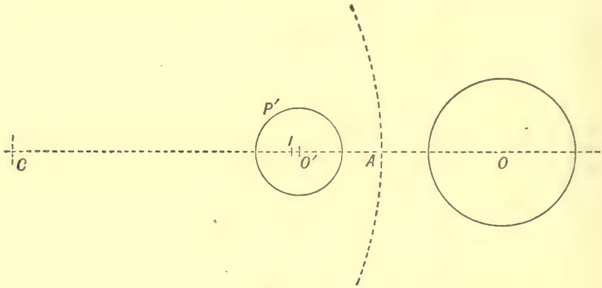

Fig. 24.

Examples.

As examples we shall invert (1) a uniformly charged sphere, (2) the induced distribution of two infinite

parallel planes under the influence of a single point-charge between them.

(1) Let the potential of the sphere be V; its radius β; the radius of inversion CA, a; the distance of C from any point P' (Fig. 24) of the image r', from the centre O' of the inverse sphere f; and the radius of the inverse sphere α. We have

(1) Uniform Spherical Distribution inverted.

$$\beta = \pm \frac{a^2 \alpha}{f^2 - \alpha^2},$$

according as C is external or internal to the given sphere. But

$$\sigma' = \frac{a^3}{r'^3}\sigma = \frac{a^3}{r'^3} \cdot \frac{V}{4\pi\beta}.$$

Substituting the value of β just found we get

$$\sigma' = \pm \frac{f^2 - \alpha^2}{4\pi\alpha} \cdot \frac{Va}{r'^3} \quad . \quad . \quad . \quad . \quad . \quad (a)$$

according as C is external or internal.

According as C is external or internal to the given sphere, and is therefore external or internal to the inverse, the spaces external and internal to the former are respectively the spaces external and internal or internal and external to the latter. Hence, according as C is external or internal, the potential of the inverse distribution at every internal point or at every external point is the same as that of a charge Va at C.

Also since the potential of the given sphere is the same for all external points as if its charge, $V\beta$, were concentrated at the centre O, the potential of the inverse distribution is, by (89), the same at every point,

Induced Distribution under Point-Charge.

external to the inverse sphere when C is external, and internal when C is internal, as that of a charge $q' = V\beta . a/CO$ concentrated at the image I of the centre of the given sphere. But $CO = \pm a^2 f/(f^2 - \mathbf{a}^2)$, and $\beta = \pm a^2 \mathbf{a}/(f^2 - \mathbf{a}^2)$, according as C is external or internal. Hence

$$q' = \frac{a}{f} . Va \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (b)$$

that is, the charge is the image in the inverse sphere of the charge $- V\mathbf{a}$ at C.

For CI we have $CI . CO = a^2$, or

$$CI = \pm \frac{f^2 - a^2}{f} . \quad . \quad . \quad . \quad . \quad . \quad . \quad (c)$$

that is, I is the image of C in the inverse sphere.

The results (a), (b), (c) are those obtained above, pp. 78, 79, 80.

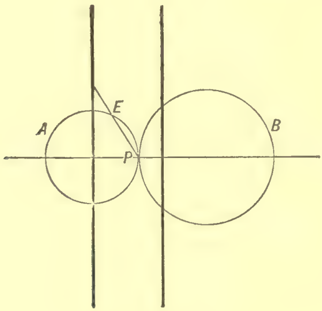

FIG. 25.

(2) Let the centre of inversion be P (Fig. 25), the radius of inversion unity, and let the planes and the

successive images be inverted, omitting the charge at P. The inverses of the planes are then spheres touching at P, and the distribution on either sphere is the inverse of the distribution on the corresponding plane. For the charges and distances of the inverse charges corresponding to the trail of images I_1, I_2, &c. (Fig. 21) we have

Inversion of Induced Distribution on Two Parallel Infinite Planes.

$$\begin{array}{lcc}
\text{Images} & I_{2n-1}, & I_{2n}, \\
\text{Charges} & \dfrac{-qa}{2(n-1)(\alpha+\beta)+2\beta}, & \dfrac{+qa}{2n(\alpha+\beta)}, \\
\left.\begin{array}{r}\text{Distances}\\ \text{from } P\end{array}\right\} & \dfrac{a^2}{2(n-1)(\alpha+\beta)+2\beta}, & \dfrac{a^2}{2n(\alpha+\beta)},
\end{array}$$

where n has every positive integral value from 1 to ∞.

The table for the images J_1, J_2, is formed from this by interchanging α and β.

The diameters of the spheres A, B are respectively $a^2\alpha$, $a^2\beta$, and therefore the inverse charges corresponding to I_1, I_2, &c., are within the sphere B, and the other series within the sphere A.

Since the potential at any point on the planes or behind them is zero, the potential at any such point due to the induced distribution on the planes is equal to that of a charge $-q$ situated at P, that is $-q/r$, where r is the distance of the point from P. The potential at any point on or within the spheres is therefore $-q/a$, a constant quantity. Again, since the potential produced by the electrification of either plane at any point on the plane or in front of it is the potential due to the trail of images behind the plane, the potential at any point

Equilibrium Distribution on Two Spheres in Contact.

Two Spheres in Contact.

on or external to either sphere produced by the distribution on the sphere is the potential at that point of the trail of images within the sphere. It follows that the charge on each sphere is equal to the sum of the image-charges whose positions fall within it; and that the distribution thus found is the equilibrium distribution when the spheres are freely electrified in contact.

Charge on each Sphere.

Denoting the charge on the sphere B by Q_B, and the radii of the spheres A, B by r_1, r_2 respectively, summing the image-charges, and substituting in the result V for $-q/a$, $a/2r_1$ for α, $a^2/2r_2$ for β, we get

$$Q_B = Vr_1r_2 \sum_{=0}^{n=\infty} \left\{ \frac{1}{r_1 + n(r_1+r_2)} - \frac{1}{(n+1)(r_1+r_2)} \right\}$$

$$= V \frac{r_1 r_2^2}{r_1 + r_2} \sum_{n=0}^{n=\infty} \frac{1}{(n+1)\{r_1 + n(r_1+r_2)\}} \qquad (90)$$

where n has every positive integral value from 1 to ∞.

A similar expression with r_1, r_2 interchanged holds for Q_A.

The capacities of the spheres are of course obtained by dividing Q_B and Q_A by V.

Electric Density at any Point.

Multiplying the expression (p. 89) for the density at any point of the plane A by a^3/r'^3, where r' is the distance from P of the corresponding point E on the sphere A, making the above substitutions, and, besides, putting $a^4(1/r'^2 - 1/4r_1^2)$ for γ^2, we get for the density at E

$$\sigma = \frac{2Vr_1^2r_2^2}{\pi} \sum_{n=0}^{n=\infty} \left[\frac{(2n+1)r_2 + 2nr_1}{[r'^2\{(2n+1)r_2 + 2nr_1\}^2 + 4r_1^2r_2^2 - r'^2r_2^2]^{\frac{3}{2}}} \right.$$

$$\left. - \frac{(2n+1)r_2 + 2(n+1)r_1}{[r'^2\{(2n+1)r_2 + 2(n+1)r_1\}^2 + 4r_1^2r_2^2 - r'^2r_2^2]^{\frac{3}{2}}} \right] \qquad (91)$$

This expression is convergent, and, when the value of r is not infinitely small, the density at any point can be approximately calculated from the potential, the radii of the spheres and the distance of the point from P. Electric Density at any Point.

For $r' = 0$ the expression (91) fails, but since the resultant force acts outwards at any point on either sphere at right angles to the surface, it follows that it must be zero at any point infinitely near the point of contact; and hence the density there must be zero. The density is a maximum on each sphere at the point diametrically opposite to the point of contact.

The density at any point on B is obtained as before by interchanging r_1 and r_2.

The charge on each sphere is the difference of the sums of two harmonic series, and cannot be obtained in finite terms. Each series is divergent, but the two taken together as in (90) constitute a convergent series, and hence the charge can be approximately calculated for given values of V, r_1, r_2. The expressions agree with those given in a different form by Poisson, who first attacked this problem, and solved it by a direct but recondite method.* It is interesting to note that equations (90) and (91) give the algebraical theorem that the integral of the expression

* *Mémoires de l'Institut*, 1r partie, p. 1. See also Plana, *Mémoires de l'Académie des Sciences de Turin*, Série II. t. vii. p. 71, for a fuller development of Poisson's method and results. A sketch of the method is given in Mascart's *Traité d'Électricité Statique*, t. i. p. 272. It is easy to convert the expression for Q_B in (90) into a definite integral agreeing with the result given by Poisson for this case.

Charge in Case of two Equal Spheres.

on the right of (91) taken over A is the expression on the right of (90) with r_1, r_2 interchanged.

When $\alpha = \beta$, $r_1 = r_2$, and we have

$$Q_A = Q_B = Vr_1 \left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \&c.\right)$$
$$= Vr_1 . \log_e 2 = 693147 . Vr_1 \quad . \quad (92)$$

or the charge on each sphere is to the charge of the sphere when alone in the field and at potential V, as $\log_e 2$ is to 1.

Case of One Sphere very Small:

If r_1 be small in comparison with r_2,

$$Q_B = \frac{Vr_2{}^2}{r_1 + r_2} + \frac{Vr_1 r_2}{r_1 + r_2} \sum_{n=0}^{n=\infty} \frac{1}{n(n+1)} = Vr_2, \quad (93)$$

since $\sum \frac{1}{n(n+1)} = 1$. The charge is therefore, nearly, the free charge which the sphere would have if alone and at potential V. The mean density is $V_2/4\pi r_2$.

Ratio of Mean Densities.

On the same supposition we have

$$Q_A = \frac{Vr_1}{r_2} \sum_{n=0}^{n=\infty} \frac{1}{(n+1)^2} = \frac{Vr_1{}^2}{r_2} \frac{\pi^2}{6}, \quad . \quad (94)$$

and the mean density is $\frac{1}{6} V\pi^2/4\pi r_2$. Thus the mean density of the small sphere is to that of the large sphere as $\pi^2/6$ (= 1·645) is to 1.

Hence when a small conducting ball is brought into contact with a large electrified spherical conductor so as to receive a charge which can then be removed on the ball and measured, the electric density at the point before contact is to the mean density on the ball as 1 is to 1·645. The same result will hold whether the conductor

be spherical or not, provided its curvature be continuous round the point of contact over a distance great in comparison with the radius of the ball, and of amount small as compared with that of the ball.

Problems Soluble by Inversion.

Many other interesting examples of the power of the method of images are to be found in the solution of the problem of two concentric uninsulated spherical conductors under the influence of a point-charge between them, and the derivation by inversion of the induced distribution on two mutually influencing spheres; the solution of the latter problem by a direct application of images and Murphy's principle of successive influences; the determination either directly or by inversion from the result of p. 90 [eqs. (85) and (85 *bis*)] of the distribution on two spheres which cut one another at right angles; the distribution on a spherical bowl,* and other important cases. For these the reader is referred to the works mentioned above, p. 99.

* For the spherical bowl see Thomson's *Electrostatics and Magnetism*, p. 178, which contains the famous original solution of the problem. The subject has recently been considered by Mr. E. G. Gallop (*Quarterly Journal of Mathematics*, No. 83, February 1886, p. 229) in an essay which merits special attention as containing the most exhaustive treatment which the problem has yet received.

Section V.

ANALOGY BETWEEN THEORY OF HEAT CONDUCTION AND THEORY OF ELECTROSTATICS.

SPECIFIC INDUCTIVE CAPACITY.

SOLUTIONS FOR SIMPLE CASES OF DIFFERENT MEDIA IN ELECTRIC FIELD.

Analogy between Theory of Heat Conduction and Theory of Electrostatics.

A remarkable analogy between the theory of the conduction of heat in solid bodies and the theory of Statical Electricity was pointed out by Sir William Thomson,* and made by him a means of establishing many of the most important theorems of electrostatics. According to the theory of heat conduction given by Fourier, the flow of heat per unit of area per unit of time (which we here call *flux of heat*), across a plane slab of homogeneous material the opposite faces of which are maintained at two different uniform temperatures, depends only on the difference of temperatures between the faces, the distance of the faces apart, and the nature of the material; and takes place in the direction from the surface of higher temperature to that of lower temperature. Thus, if the thickness of the slab be x centimetres and v, v' the uniform temperatures of its faces, the quantity of heat, Q, conducted in a second of time from the face of temperature v to that of temperature v', across one square centimetre, is given by the equation

Flux of Heat defined.

$$Q = k\frac{v - v'}{x}, \quad . \quad . \quad . \quad . \quad . \quad (95)$$

* Reprint of Papers on Electrostatics and Magnetism, p. 1.

where k is a constant (always positive) depending on the material. From the fact that there is the same flux of heat inwards across one face that there is outwards across the other, and therefore also across all intermediate parallel sections, it follows that the variation of temperature from one face to the other is uniform.

Flux of Heat defined.

The constant k is called the *Thermal Conductivity* of the substance. In bodies which have a different molecular structure in different directions (that is which are not *isotropic* * in structure), such as unannealed glass, or a piece of otherwise isotropic glass distorted by flexure or by pressure unequally applied to its surface, all crystals except those of the cubic system, fibrous materials, &c., the value of k is also different in different directions. We shall consider first the flow of heat in a solid body for which the value of k is the same in all directions and at all points.

Thermal Conductivity defined.

Let the temperature vary continuously over each of two parallel sections in the body, and from one section to the other, though not necessarily with a uniform gradient. If the sections be taken very close together, the deviation of the temperature gradient from uniformity along any line drawn normally from one to the other may be neglected, and if a sufficiently small area be taken in each section at the extremity of that straight line, the deviation from uniformity of temperature over that area may also be neglected. Taking equal areas in the two

Flow of Heat in a Homogeneous Isotropic Solid.

* A body in which any given quality is at all points the same in all directions is said to be *isotropic* as to that quality. The term *isotropic* was introduced by Cauchy; the term *æolotropic*, from *αἰόλος*, *variegated*, has been used by Thomson and Tait as the negative of isotropic. A body may evidently be *homogeneous*, that is equal cubical portions in different parts of the body which have their sides in the same directions may be precisely similar and yet the body may be æolotropic.

Measure of Heat at any point in Isotropic Solid.

sections, each thus small in any dimension, and at the same time large in comparison with the distance of the sections apart, we may neglect the effect of the remainder of the sections, and consider only the flux from one small area to the other, and apply the result obtained above. Denoting each area by ds, and their distance apart by dx, and putting dQ for the quantity of heat conducted from one to the other, we have

$$dQ = -k\frac{dv}{dx}ds \quad . \quad . \quad . \quad . \quad . \quad . \quad (96)$$

In the limit, therefore, we have for the flux in any direction, x, at a point in a solid at which the temperature is v, the value $-k\frac{dv}{dx}$, and the flow per unit of time across an element of area ds at right angles to the direction of x and passing through the point is $-k\frac{dv}{dx}ds$.

Motion of Heat in Isotropic Solid.

Now consider a small parallelepiped of the material with faces parallel to three rectangular axes of x, y, z, and bounded by edges of lengths dx, dy, dz. Let the temperature at the centre of the parallelepiped be v, then the temperatures at the centres of the y, z faces are

$$v - \tfrac{1}{2}\frac{dv}{dx}dx, \quad \text{and} \quad v + \tfrac{1}{2}\frac{dv}{dx}dx.$$

The total flow from left to right across the left-hand y, z face is therefore

$$-k\frac{d}{dx}\left(v - \tfrac{1}{2}\frac{dv}{dx}dx\right)dy\,dz,$$

Differential Equation found.

and across the opposite face in the same direction is

$$-k\frac{d}{dx}\left(v + \tfrac{1}{2}\frac{dv}{dx}dx\right)dy\,dz.$$

The excess of the outflow above the inflow in the direction of x is therefore

$$-k\frac{d^2v}{dx^2}dx\,dy\,dz.$$

Proceeding in the same way for the other two pairs of opposite sides, we get for the total excess of the outflow above the inflow the value

$$-k\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right)dx\,dy\,dz.$$

If this have a positive value there is a generation of heat within the prism; if a negative value more flows in than flows out. In the latter case if c be the thermal capacity of the solid per unit of volume, the thermal capacity of the parallelepiped is $c\,dx\,dy\,dz$. Dividing by this the last found expression with the sign changed we get the time-rate of rise of temperature of the element, or

Differential Equation found.

$$\frac{dv}{dt} = \frac{k}{c}\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) \quad . \quad . \quad . \quad . \quad (97)$$

If a rise of temperature in the substance of the prism is not to result the excess must be carried off or transformed within the element itself. If the outflow is equal to the inflow there is neither generation nor absorption of heat, and no change occurs in the temperature of the prism. Including the case of generation of heat, and putting $4\pi\mu\,dx\,dy\,dz$ for the heat generated within the prism per unit of time, so that $4\pi\mu$ is the mean value over the prism of the time rate of generation of heat per unit of volume, we have when there is no change of temperature in the element

$$k\left(\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}\right) + 4\pi\mu = 0 \quad . \quad . \quad . \quad (97a)$$

In the same way if $dv_1/d\nu_1$, $dv_2/d\nu_2$ be the rate of variation of temperature along normals drawn into two media from a point in a surface separating them, and if k_1, k_2 be the conductivities of the respective media, the quantity of heat conducted away from the surface along the normal per unit of time per unit of area are $-\,k_1dv_1/d\nu_1$, $k_2dv_2/d\nu_2$, and this must be equal to the time rate of generation of heat at the surface per unit of area. Denoting the latter by $4\pi\mu$, we have the surface equation

$$k_1\frac{dv_1}{d\nu_1} + k_2\frac{dv_2}{d\nu_2} + 4\pi\mu = 0. \quad . \quad . \quad . \quad . \quad (97b)$$

Thus if we consider the case of a distribution of constant heat sources and a corresponding system of temperatures in the conducting medium surrounding them which also remain constant with the time, that is, the case in which sources have existed and given out heat so long at a constant rate that the medium has attained a permanent state as to temperature, we obtain an

Analogy of Theory of Steady Motion of Heat

equation which corresponds exactly to Poisson's equation, and holds for all volume-sources, positive or negative, that is where heat is supplied to the body or carried off, and corresponds to Laplace's equation at all other places, for there $\mu = 0$. A surface differential equation (97b) corresponding exactly to (26) above has also been found, by an exactly similar process, for a surface distribution of thermal sources, and we have seen that in this case the rate of generation of heat per unit of area corresponds to electric surface density multiplied by 4π. Hence taking temperature as the analogue of electric potential, and rate of generation of heat per unit of volume, or per unit of area, as the analogue of electric volume density or electric surface density, multiplied by 4π, the fundamental differential equations are the same in the theory of heat conduction in a body in a permanent state, and in that of a system of electric charges surrounded by an insulating medium. Further, since to a given system of sources corresponds one, and only one, system of temperatures at all points of the conducting medium, and these temperatures must be zero at all points infinitely distant from the source, the electrical and thermal theories are identical, and every solution of a problem in one is capable of being interpreted as the solution of a problem in the other; that is, to a given distribution of heat sources and the resulting temperatures in a conducting medium surrounding them, corresponds an exactly similar distribution of electric charges and their resulting potentials at different points in an insulating medium, and conversely. Many electrical theorems are thus capable

to Electrostatic Theory.

Analogues in the two Theories.

Results mutually interpretable.

of being established as analogues of known, and indeed obvious, thermal results.

For example, consider a single constant point-source of heat in a medium of conductivity k in all directions, and let the total heat generated at the source in unit of time be Q. Plainly this quantity of heat must flow out of every closed surface described round the source. By what has been shown (p. 104) above, we have, for the total flux across a spherical surface of radius r having the source at its centre, the value

Examples: Single Point-source in Isotropic Medium

$$- 4\pi r^2 k \frac{dv}{dr} = Q,$$

and therefore

$$- \frac{dv}{dr} = \frac{Q}{4\pi k} \cdot \frac{1}{r^2}.$$

Hence integrating we find

identical with Single Point-charge in Isotropic Dielectric.

$$v = \frac{Q}{4\pi k} \cdot \frac{1}{r} + C,$$

when $r = \infty$, $v = 0$, and therefore $C = 0$. Hence

$$v = \frac{Q}{4\pi k} \cdot \frac{1}{r}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

that is the temperature is inversely as the distance from the centre. Putting $k = 1$, we see that v is also the potential at a point at a distance r from an electric charge $Q/4\pi$ concentrated at a point.

Superposition of Temperatures and Fluxes.

The principle of superposition holds for temperatures as well as for potentials, and we have for the temperature at any point, and for the flux of heat at any point and in any direction, the sum of the temperatures and the sum of the fluxes in the particular direction due to the several sources separately. Now consider the flow of heat across any closed surface inclosing part of a constant system of sources of heat in a medium which has attained the permanent state. All

Thermal Analogue of Electric Induction.

the heat which flows in across the surface from sources without the surface must flow out again elsewhere, and the whole heat generated within the surface must flow out across it. Hence if $-kdv/d\nu$ be the flux of heat along the normal outwards across the surface at any element ds, the total flow outwards across the whole surface produced by the whole system of sources is $\iint -k\,\dfrac{dv}{d\nu}\,ds$ taken over the surface, and this must be equal to the whole quantity of heat generated within the surface. We have therefore a theorem for heat conduction corresponding to (13) of p. 12 above.

Isothermal Surfaces.

Surfaces of equal temperature, or *Isothermal Surfaces*, correspond to equipotential surfaces in the electrical analogue, and the flow of heat across any such surface is at every point along the normal. Now it is evident that the temperature at any point without any isothermal surface inclosing all the sources is independent of the position of the sources within it, provided the temperature of the surface remains unchanged; and we may therefore suppose the sources distributed over the surface so as to fulfil this condition. If this be done the temperature at every point within the isothermal surface must be the same as at the surface; for, as there are no sources within the surface, the flow outwards from within is on the whole zero, and therefore the total flow from the external isothermal surface within is zero, and so on for successive isothermal surfaces within. Hence there is no flow of heat anywhere within, that is, the temperature is constant.

If then v be the temperature at the surface, and $4\pi\mu$

the amount of heat generated in unit of time per unit area at any element ds of the surface, the temperature at any point at distance r from ds is, by the result found (p. 107) above, $\mu . ds/kr$. Hence the total temperature at any point is $\frac{1}{k}\iint\frac{\mu}{r}\,ds$ taken over the surface, and this must be equal to v for any point either on or within the isothermal surface.

Distribution of Sources over Isothermal with Intensity equal to Flux

Now since the temperature of the surface and the external temperature remain unchanged the flux of heat at every external point remains unchanged. Hence the flux from the surface outwards is $-kdv/dv$ at each point as before, and since the flux is normal to the surface this must be equal to the value of μ at the point in question. We have therefore the result that a distribution of sources of heat over the isothermal surface such as to give intensity equal to $-kdv/dv$ at each point will produce the same external system of temperatures. From this we get, putting $k=1$, the important electrical theorem proved in pp. 29 and 73 above.

Equivalent to Actual Sources.

So far we have considered the medium surrounding the conductors to be vacuum, and we have defined unit quantity of electricity on this supposition as that quantity of electricity which placed at unit distance from an equal quantity would be repelled with unit force. But in all cases, even in a so-called vacuum, say the most complete that can be made by the best air-pump on the Sprengel or Geissler principle, there must, according to the theory proposed by Faraday and now almost universally accepted, be a medium or *dielectric* which transmits the electric influence. To this view

Influence of Dielectric Medium.

Influence of Dielectric Medium.

Faraday was led by the results of his own experiments, which proved that the phenomena of electrostatic induction depend on the nature of the dielectric medium interposed between the conductors; and he gave a theory of the action of the medium which has since been fully worked out mathematically by Thomson, Maxwell, and others. It would be out of place here to give details of the theoretical investigations of these writers; we shall merely state briefly such results as for the most part will be of use to us in the accounts of measurements which follow.

Specific Inductive Capacity of a Dielectric.

The consideration of the propagation of electric action through a medium, as due to a polarization of its particles along the lines of force so that each becomes oppositely electrified at its extremities, shows that the transmission of electric force depends on a certain characteristic property of the medium, in precisely the same way as the rate of flow of heat in a substance depends on the thermal conductivity of the material. We have, in the sketch of the analogy between heat conduction and electrostatic theory above, put $k = 1$, in interpreting electrically the results. The behaviour of different media can however be accurately expressed by supposing each medium to have at every point a quality which is the analogue of thermal conductivity in the parallel theory, and is called the *Specific Inductive capacity* of the medium. We shall see when we come to deal with magnetism that precisely similar considerations apply to magnetic media. Also we shall see, chap. ii., that the Theory of Flow of Electricity is completely analogous to that

of Heat Conduction, or that of the motion of an incompressible fluid under certain conditions.

As most of the experimental investigations hitherto made have been on media which have the same electrical qualities in all directions, we shall give here only some general considerations regarding such cases, and shall deal specially with such questions regarding crystals and other bodies of æolotropic structure as may arise.

Modification of Theoretical Results for Medium of Sp. Ind. Cap. K.

The thermal analogy shows clearly how the results given above for a medium of unit specific inductive capacity (or *vacuum* as we denote such a medium for brevity), are to be modified in the case of any other medium. We have seen that the total flow of heat across any closed surface in a medium of thermal conductivity k is $-\int\int k ds \,.\, dv/dv$. Denoting the specific inductive capacity by K, we have for the total quantity of electricity within any closed surface in the electric field the value $-\frac{1}{4\pi}\int\int K\frac{dV}{d\nu}ds$; and $-Kds\,.\,dV/d\nu$ is now the *Electric Induction* across the element ds, $-dV/d\nu$ as before the component electric force at right angles to the surface. The induction is thus a directed quantity which has at every point in an isotropic medium the same direction as the electric force.

We have thus in all the investigations above to substitute for the component forces $-dV/dx$, &c., the expressions $-KdV/dx$, &c. (which are called the *components of Electric Induction*), to make the results applicable to the case of an isotropic medium of inductive capacity K. For example in Green's Theorem, p. 63 above, we obtain the equation of energy by putting $U = V$, and $a^2 = K$, and dividing by 8π.

Force between two Point-charges

We also see that the force between two quantities q, q' of electricity concentrated at points at distance r apart in such a medium is qq'/Kr^2.

It may be noticed here that just as in the thermal analogy the direction of the flux of heat in an æolotropic body is not in general at right angles to the isothermal surfaces, so in an æolotropic medium the direction of the resultant electric induction at any point is not in general the same as the direction of the resultant electric force at the same point. Here also the

Modified Form of Poisson's Equation.

two theories are parallel, but it is beyond our limits to enter into them.

We also get at once the modified characteristic differential equation for a medium of specific inductive capacity K, varying from point to point, but the same in all directions at any one point,—

$$\frac{d}{dx}\left(K\frac{dV}{dx}\right) + \frac{d}{dy}\left(K\frac{dV}{dy}\right) + \frac{d}{dz}\left(K\frac{dV}{dz}\right) + 4\pi\rho = 0, \quad (99)$$

and at any electrified surface in the medium,—

$$K\left(\frac{dV_1}{d\nu_1} + \frac{dV_2}{d\nu_2}\right) + 4\pi\sigma = 0. \quad . \quad . \quad . \quad . \quad (100)$$

Surface Equations.

If the electrified surface be a surface of separation between two media of specific inductive capacities K_1, K_2, the surface equation is by (97*b*) above

$$K_1\frac{dV_1}{d\nu_1} + K_2\frac{dV_2}{d\nu_2} + 4\pi\sigma = 0. \quad . \quad . \quad . \quad (101)$$

Case of several Media in same Field.

In the case of a field occupied in different regions by media of different specific inductive capacities, the characteristic equation is to be applied with the corresponding value of the K's in each region, and the surface equation (101) at each separating surface.

"True" and "Apparent" Electrifications.

It is to be observed that the electric densities ρ and σ are the true electric densities which exist in the form of an electric charge conveyed to the medium or placed on the surface, and do not include the electrification of the medium in consequence of induction.

We may put the theorem of (101) into words as follows.

If N_1, N_2 be the normal forces at infinitely near points on opposite sides of the surface of separation between two isotropic media, each force being reckoned in the direction from the surface, K_1, K_2, the specific inductive capacities of the respective media, and if there is no electric charge on the surface except that due to induction, then

$$K_1N_1 + K_2N_2 = 0. \quad . \quad . \quad . \quad . \quad (101\ bis)$$

This equation may be written in the form

Case of several Media in same Field. "True" and "Apparent" Electrifications.

$$N_1 + N_2 - 4\pi\sigma' = 0, \quad . \quad . \quad . \quad (102)$$

where

$$\sigma' = \frac{1}{4\pi} \cdot \frac{K_2 - K_1}{K_2} N_1 = \frac{1}{4\pi} \cdot \frac{K_1 - K_2}{K_1} N_2. \quad (103)$$

This value of σ' is the electric surface density which would exist on the separating surface of the media if each had unit specific inductive capacity and N_1, N_2 their actual values, and has been called by Maxwell * the *apparent* electric density on the surface. If a distribution of this density be made over the surface of the space occupied by K_2, and the specific inductive capacities K_1, K_2 be made each unity, the same electric force will be produced at all points internal or external. For the distribution if made gives the actual values of N at the surface, and equation (99) will plainly be satisfied; and we have seen that under these conditions there can be only one value of the potential at any point.

If this apparent electrification be removed during the action of the inducing force by bringing every part of the surface to zero potential, say, by passing a flame over it, and the inducing force be then removed, there will appear a true electrification equal and opposite to σ'. This fact has been used by Sir William Thomson to explain the phenomena of pyro-electricity shown by certain crystals.†

We may also write for Poisson's equation, (99) above,

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + 4\pi\rho' = 0. \quad . \quad . \quad . \quad (104)$$

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* *El. and Mag.*, vol. i., 2nd ed., p. 95.
† *Ibid.* p. 58.

where

$$\frac{dK}{dx}\frac{dV}{dx} + \frac{dK}{dy}\frac{dV}{dy} + \frac{dK}{dz}\frac{dV}{dz} + 4\pi(\rho - \rho' K) = 0. \qquad (105)$$

If K is constant the last equation gives

$$\rho' = \frac{\rho}{K}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (106)$$

The value of ρ given by (106) is that required in a field of specific inductive capacity unity to produce the same potential as is produced in the actual field by the density ρ.

Refraction of Lines of Force.

If the surface of separation is not at right angles to the lines of force, then resolving the forces at two infinitely near points on opposite sides of the surface along and at right angles to the normal, we have by (101), if the surface is not electrified,

$$K_1 \frac{dV_1}{d\nu_1} + K_2 \frac{dV_2}{d\nu_2} = 0,$$

and since $V_1 = V_2$, at every point of the surface,

$$\frac{dV_1}{d\omega} = \frac{dV_2}{d\omega},$$

where $dV/d\omega$ denotes rate of variation of potential in a direction parallel to the surface of separation, and in the plane of the line of force and the normal. Hence if θ_1, θ_2 be the angles which the line of force makes with the normal in the first and in the second medium respectively, we have

$$\tan\theta_1 = \frac{dV_1}{d\omega} \Big/ -\frac{dV_1}{d\nu_1}, \quad \tan\theta_2 = \frac{dV_2}{d\omega} \Big/ \frac{dV_2}{d\nu_2},$$

and therefore

$$\tan\theta_1 = \frac{K_1}{K_2}\tan\theta_2. \quad . \quad . \quad . \quad . \quad . \qquad (107)$$

Comparison of Electrostatic and Optical Refraction.

The line of force thus undergoes a species of refraction in which the tangents of the angles of incidence and refraction are related as are the sines of the corresponding angles in the refraction of light. It is to be observed that according to the law of refraction of lines of force they can show nothing corresponding to the optical phenomenon of total reflection. This refraction is illustrated in Fig. 26.

If the surface of separation between two media be at right angles to the lines of force in one medium, it is by equation (107) at right angles to the lines of force in the other, that is, the surface is an equipotential surface.

The apparent electrification and the force at any point are easily found in the following simple cases, which it is instructive to consider. Examples.

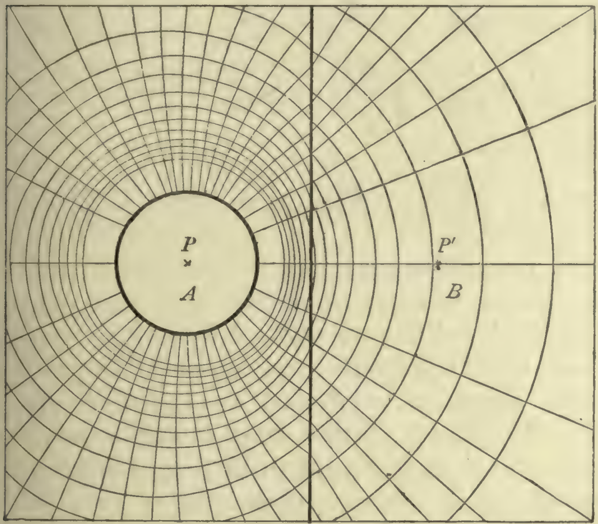

FIG. 26.

(1) Two isotropic media A, B (Fig. 26), of specific inductive capacities K_1, K_2, have a plane surface of separation, and a point-charge of amount q is situated at a point P in the medium A at a distance d from the plane of separation. The normal force due to the charge (1) Two Media separated by Plane Surface.

Two Media separated by Plane Surface.

q is at any point E in the plane, $q \,.\, d/PE^3$ in the direction from A to B. Hence if σ' be the apparent surface density at E we have the equations

$$\left.\begin{aligned} N_1 &= -\frac{qd}{PE^3} + 2\pi\sigma', \\ N_2 &= \quad \frac{qd}{PE^3} + 2\pi\sigma'. \end{aligned}\right\} \quad . \quad . \quad . \quad (108)$$

But by (103)

$$\sigma' = \frac{1}{4\pi}\,\frac{K_2 - K_1}{K_2}\,.\,N_1 = \frac{1}{4\pi}\,\frac{K_1 - K_2}{K_1}\,.\,N_2.$$

Hence we have

$$N_1 = -\frac{2K_2}{K_1 + K_2}\cdot\frac{qd}{PE^3}, \quad N_2 = \frac{2K_1}{K_1 + K_2}\cdot\frac{qd}{PE^3}. \quad (109)$$

The force at any point in the medium A is therefore by the principle of electric images (p. 84) that which, if A and B were replaced by a single medium of unit specific inductive capacity, would be produced by the charge q at P, and a charge of amount $q\,(K_1 - K_2)/(K_1 + K_2)$ at the image P' of P in the plane; and the force at any point in B is that which would be produced in the same circumstances by a charge $2q\,K_1/(K_1 + K_2)$ at P.

The directions of the lines of force (or, as they ought rather to be called, lines of induction) for the case of $K_2 = 5$, $K_1 = 3$, are shown in Fig. 26, which represents a section of the electric field made by a plane passing through P and cutting the plane of separation normally. The closed curves which surround P are sections of equipotential surfaces, and the lines cutting

them at right angles are the lines of induction. The equipotential surface nearest P in the diagram is a sphere, and the surfaces interior to it are omitted. The distances between successive equipotential surfaces on the left show how the electric induction K_1F varies from point to point in the medium, and the lines of induction are so drawn that if the diagram were rotated about the line PP' the field on the left of the plane of separation would be divided by the equipotential surfaces, and those generated by the lines of induction, into cellular spaces each containing the same amount of the electrical energy, considered, as in p. 34 above, as having its seat in the medium.

Two Media separated by Plane Surface.

On the right of the separating line the curves are the continuations of those on the left into the second medium. The equipotential surfaces are here spherical surfaces with P as centre, and the lines of induction straight lines radiating from P.

The apparent density for any given electric system is of course obtained by superimposing the densities thus found for the several elements of the system.

(2) Two Dielectrics separated by a Plane Layer of a third, under Influence of Point-Charge in one.

(2) It is easy to apply the principle of images to find the force and potential at any point, and the density of the apparent electrification, in certain simple cases of three or more media occupying different regions of the electric field. For instance, let the infinite planes AB (Fig. 27) be surfaces of separation of three dielectrics which completely occupy the field, and let the electric system be a point-charge at P in the medium between the planes. Let the specific inductive capacity of the medium to the left of A be denoted by K_1, of the

Two Dielectrics separated by a Plane Layer of a third, under Influence of Point. Charge in one

medium to the right of B by K_2, and that of the medium between the planes by K, and write μ_1 for $(K-K_1)/(K+K_1)$, μ_2 for $(K-K_2)/(K+K_2)$. The apparent electrifications of the planes A, B, can be built

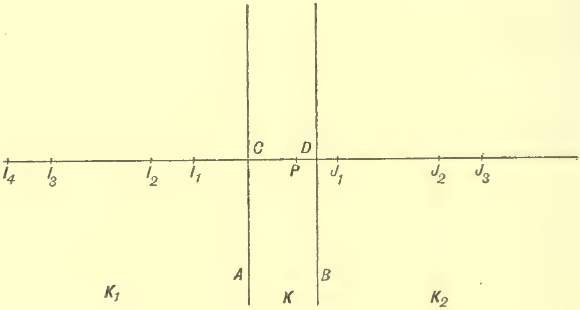

FIG. 27.

up by considering the apparent electrification on A due directly to the charge q at P, then the apparent electrification of B due to q at P and the electrification of A already found, then the electrification of A due to the electrification of B already found, and so on. We get in this manner the result that the potential, and therefore the force at any point whatever, produced by the apparent electrification, is that due to a series of charges of the amounts and in the positions specified in the following table :—

CHARGES EQUIVALENT TO APPARENT ELECTRIFICATION.

Apparent Electrification of Planes—(1) When Charge is in Middle Stratum :

I.—Plane A.

Positions (Fig. 27).

P or I_1,	J_1 or I_2,	J_2 or I_3,	J_3 or I_4, &c.

Charges.

$\mu_1 q$,	$\mu_1\mu_2 q$,	$\mu_1^2\mu_2 q$,	$\mu_1^2\mu_2^2 q$, &c.

Distances from plane A.

a,	$a + 2\beta$,	$3a + 2\beta$,	$3a + 4\beta$, &c.

Total charge on plane $A = \dfrac{\mu_1(1+\mu_2)q}{1-\mu_1\mu_2}$.

The &c.s here indicate that the series are to be continued to infinity according to the law indicated.

II.—Plane B.

The same table with I written for J, J for I, μ_1 for μ_2, and μ_2 for μ_1, a for β, and β for a throughout.

In these tables alternative positions are given for the charges. Of these the first in each case in the first table and the second in each case in the second table, or *vice versâ*, are to be used in calculating the potential (V_1 or V_2) at any point according as the point is on the left of A or the right of B, and the second in each case in both tables in calculating the potential (V) at any point between A and B. If Q be any point, we have for the potential in each of the three cases the values—

Apparent Electrification of Planes—

$$\left.\begin{aligned}V_1 = q\,\frac{1+\mu_1}{QP} &+ \\ q\,(1-\mu_1)\,\mu_2\left(\frac{1}{QJ_1}+\frac{\mu_1}{QJ_2}+\frac{\mu_1\mu_2}{QJ_3}+\frac{\mu_1{}^2\mu_2}{QJ_4}+\&c.\right)&;\\ V = \frac{q}{QP}+\mu_1 q\left(\frac{1}{QI_1}+\frac{\mu_2}{QI_2}+\frac{\mu_1\mu_2}{QI_3}+\&c.\right)&\\ +\mu_2 q\left(\frac{1}{QJ_1}+\frac{\mu_1}{QJ_2}+\frac{\mu_1\mu_2}{QJ_3}+\&c.\right)&.\end{aligned}\right\}\quad(110)$$

The expression for V_2 is obtained from V_1 by writing μ_1 for μ_2, μ_2 for μ_1, and I for J throughout.

(2) When Charge is in one of the Side Dielectrics.

Let P, the position of the charge, be in the medium to the left of A, and let in this case K, K_1, K_2 denote

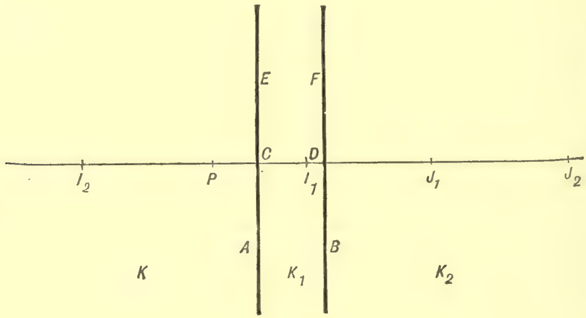

FIG. 28.

the specific inductive capacities of the media in the order from left to right. We have the following table in which $\mu_1=(K-K_1)/(K+K_1)$, $\mu_2=(K_1-K_2)/(K_1+K_2)$:—

Apparent Electrification of Planes—when Charge is in one of the side Dielectrics.

Charges equivalent to Apparent Electrification.

Plane A.

Positions (Fig. 28).

I_1 or P, J_1 or I_2, J_2 or I_3, J_3 or I_4.

Charges.

$\mu_1 q$, $-(1+\mu_1)\mu_1\mu_2 q$, $(1+\mu_1)\mu_1^2\mu_2^2 q$, $-(1+\mu_1)\mu_1^3\mu_2^3$, &c.

Distances from plane A.

a, $2\beta - a$, $4\beta - 3a$, $6\beta - 5a$, &c.

$$\text{Total charge on plane } A = \frac{\mu_1(1-\mu_2)q}{1+\mu_1\mu_2}.$$

Plane B.

Positions (Fig. 28).

P or J_1, I_2 or J_2, I_3 or J_3, I_4 or J_4, &c.

Charges.

$(1+\mu_1)\mu_2 q, -(1+\mu_1)\mu_1\mu_2^2 q, (1+\mu_1)\mu_1^2\mu_2^3 q, -(1+\mu_1)\mu_1^3\mu_2^4$, &c.

Distances from plane B.

β, $3\beta - 2a$, $5\beta - 4a$, $7\beta - 6a$, &c.

$$\text{Total charge on plane } B = \frac{\mu_2(1+\mu_1)q}{1+\mu_1\mu_2}.$$

If V, V_1, V_2 be as before the potentials at a point Q in the media of specific inductive capacities K, K_1, K_2, respectively, we have—

Apparent Electrification of Planes—

$$\left.\begin{aligned}
V_1 &= q\left(\frac{1}{QP} + \frac{\mu_1}{QI_1}\right) \\
&\quad + (1-\mu_1^2)\,\mu_2\left(\frac{1}{QJ_1} - \frac{\mu_1\mu_2}{QJ_2} + \frac{\mu_1^2\mu}{QJ_3} - \&c.\right); \\
V_2 &= q(1+\mu_1)(1+\mu_2)\left(\frac{1}{QP} - \frac{\mu_1\mu_2}{QI_2} + \frac{\mu_1^2\mu_2^2}{QI_3} - \&c.\right); \\
V &= q\,(1+\mu_1)\left\{\frac{1}{QP} - \frac{\mu_1\mu_2}{QI_2} + \frac{\mu_1^2\mu_2^2}{QI_3} - \&c.\right. \\
&\quad \left. + \mu_2\left(\frac{1}{QJ_1} - \frac{\mu_1\mu_2}{QJ_2} + \frac{\mu_1^2\mu_2^2}{QJ_3} - \&c.\right)\right\}.
\end{aligned}\right\} \quad (111)$$

The apparent density at any point on either plane is easily found from the charges and their distances in the manner shown in pp. 84, 89 above.

(3) Dielectric Sphere in Uniform Field.

(3) The following case is of great importance in the theory of magnetism and of practical interest in the experimental determination of specific inductive capacities. A spherical portion of an isotropic dielectric medium in which the electric force has everywhere the same magnitude and direction, that is, in which there is a uniform field of force, is replaced by an equal spherical portion of another isotropic dielectric. It is required to find the apparent electrification, and thence the force at any point without or within the sphere.

Let K_1, K_2 be the specific inductive capacities of the surrounding medium and the sphere respectively, F the uniform electric force in the first medium produced independently of the apparent electrification, N_1, N_2 the external and internal normal component forces at any point due to the apparent electrification, σ' the surface

density of the apparent electrification at that point of the separating surface, and θ the angle which a radius drawn to the point makes with the positive direction of F. Taking N_1, N_2 in the direction from the surface on both sides we get by (101 *bis*) and (102)

Condition satisfied by Normal Forces at Surface.

$$F\cos\theta + N_1 + \frac{K_2}{K_1}(-F\cos\theta + N_2) = 0\,;$$

or

$$\sigma' = -\frac{1}{4\pi}\cdot\frac{K_1 - K_2}{K_2}(F\cos\theta + N_1)$$

$$= -\frac{1}{4\pi}\cdot\frac{K_2 - K_1}{K_1}(-F\cos\theta + N_2). \quad . \quad (112)$$

This is the surface characteristic equation. The distribution supposed formed in the following manner

Apparent Density given by "Couches de Glissement."

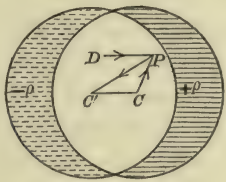

FIG. 29.

satisfies this equation at the surface, and Laplace's equation at every internal and external point, and gives therefore the apparent surface density for the case. Two equal spherical volume distributions of electricity of uniform density ρ, one positive, the other negative, and of the same radius as the sphere, are placed in coincidence; then, according as K_2 is greater or less than K_1, the positive or the negative distribution

is displaced, Fig. 29, in the direction of F through a finite distance a less than the sum of the radii. A positive volume distribution of meniscus shape is thus formed on one side, a negative distribution precisely similar on the other, and in the space occupied by the coincident parts of the distributions there is zero electric density. Now let the distance a be diminished indefinitely and the density of the volume distribution ρ increased so that ρa remains equal to its former value.

Law of Density.

Drawing then any radius making an angle θ with the direction of F, we have for the thickness of the stratum in the direction of the radius the value $a \cdot \cos\theta$. Hence, writing σ'_0 for $a\rho$, the surface density at the extremity of the radius is $\sigma'_0 \cos\theta$. Its value is σ'_0 or $-\sigma'_0$ according as $\theta = 0$, or $180°$.

Force at any Internal Point due to Distribution.

The force at any internal point P due to the distribution is plainly the resultant of the two forces due to the two spherical portions of the volume distributions which have C, C' as centres and P a common point on their surfaces. These forces are in magnitude respectively $4/3 \cdot \pi\rho \cdot CP$, $4/3 \cdot \pi\rho \cdot C'P$, and act in the directions shown in the figure, and therefore their resultant acts in the direction CC'. Putting R for this resultant taken positive in the direction of F, we have

$$R = -\frac{4}{3}\pi\rho\, CC' = -\frac{4}{3}\pi\sigma'_0. \quad . \quad . \quad (113)$$

Internal Field Uniform.

It is therefore constant in magnitude. The total force, $F + R$, within the sphere, is therefore also constant in magnitude and direction.

By (113),

Internal Field Uniform.

$$N_2 = \frac{4}{3}\pi\sigma'_0 \cos\theta = \frac{4}{3}\pi\sigma',$$

which gives by substitution in (112)

$$\sigma' = \frac{3}{4\pi}\frac{K_2 - K_1}{2K_1 + K_2} F \cos\theta = \sigma'_0 \cos\theta. \quad (114)$$

Therefore

$$R = -\frac{K_2 - K_1}{2K_1 + K_2} F, \quad . \quad . \quad . \quad (115)$$

and

$$F + R = \frac{3K_1}{2K_1 + K_2} F. \quad . \quad . \quad . \quad (116)$$

Resultant Internal Force.

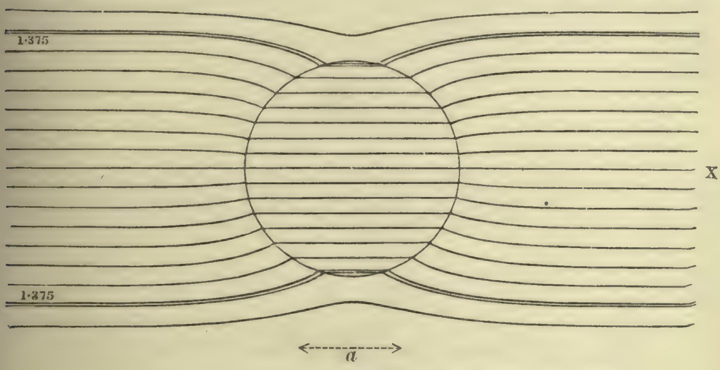

FIG. 30.

Hence according as K_2 is greater or less than K_1 the force within the sphere is less or greater than the force F without.

The directions of the lines of force outside and

Resultant Internal Force.

inside the sphere are shown in Fig. 30 * for the case of $K_2 = 2{\cdot}8K_1$, and radius of sphere $= 1{\cdot}1a$; in Fig. 31 for $K_2 = {\cdot}48K_1$ and radius of sphere $= 1{\cdot}34a$.

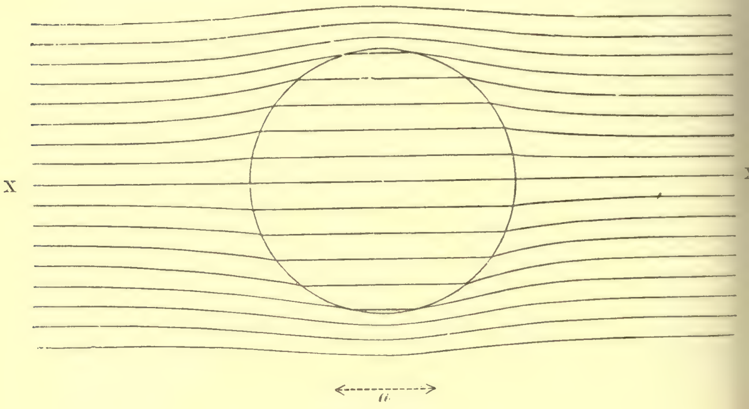

Fig. 31.

Case of Conducting Sphere in Uniform Field.

If the sphere is of conducting material, $K_2 = \infty$, and $F + R = 0$, as it ought to be. In this case also we have

$$\sigma' = \frac{3}{4\pi} F \cos\theta. \quad . \quad . \quad . \quad . \quad (117)$$

The directions of the lines of force for the case of the conducting sphere are shown in Fig. 32.* The radius of the sphere is $a/\sqrt[3]{2} = {\cdot}794a$.

* The equation of the curves external to the circle in Figs. 30, 31, 32, is

$$x = \left\{ \left(\frac{a^2y^2}{b^2 - y^2} \right)^{\frac{2}{3}} - y^2 \right\}^{\frac{1}{2}}.$$

The centre of the circle is the origin, and the curve XX, which in each

The potential energy of the dielectric sphere in the uniform field is found simply by calculating the work

Potential Energy of Dielectric

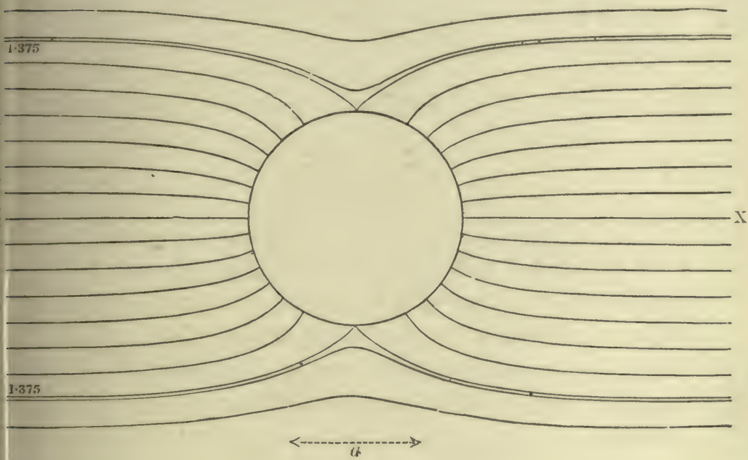

Fig. 32.

done by electric forces in the relative displacement of the imaginary volume distributions. If r be the radius

Sphere in Uniform Field.

case is a straight line, is the axis of x. In Figs. 30 and 32, y^2 is everywhere less than b^2; in Fig. 31, y^2 is everywhere greater than b^2. Each set of curves is drawn for a constant value of a which is indicated below the diagram, and values of b equal to 0, $\cdot 2a$, $\cdot 4a$, $\cdot 6a$, ... $1\cdot 6a$. In Figs. 30 and 32, the curve for $b = \sqrt{3}/\sqrt[3]{2}\,.\,a = 1\cdot 375a$ is drawn. This curve has a pair of double points through which the circle in Fig. 32 passes: in Fig. 30 these points fall within the circle and are not shown. In Fig. 32 the circle has radius $= a/\sqrt[3]{2} = \cdot 794a$ and cuts orthogonally all the curves except that on which are the double points: in Figs. 30 and 31 the radii of the circles are $1\cdot 1a$ and $1\cdot 34a$ respectively. [See Sir W. Thomson's *Reprint of Papers on Electrostatics and Magnetism*, p. 492, from which these Figures are taken.]

Potential Energy of Dielectric Sphere in Uniform Field.

of the sphere, the total quantity of electricity in the positive volume distribution is $4/3 \,.\, \pi\rho r^3$. The work done by electric forces in displacing this through a distance a is $4/3 \,.\, \pi\rho r^3 \,.\, Fa$. Hence, if E be the energy of the sphere *in the field*,

$$E = \frac{4}{3}\,\pi r^3 F \,.\, \rho a = \frac{4}{3}\,\pi r^3 F \sigma_0'$$

$$= -\, r^3\,\frac{K_2 - K_1}{2K_1 + K_2} \,.\, F^2. \quad . \quad . \quad . \quad . \quad (118)$$

This expression has been obtained for a uniform field, but it will also hold for a variable field if r be so small that the value of F is sensibly constant in magnitude and direction at every point of the sphere.

Force in Variable Field: (1) on Dielectric Sphere;

On this supposition, the rate of diminution of E in any direction ν in a variable field is given by the equation

$$-\,\frac{dE}{d\nu} = r^3\,\frac{K_2 - K_1}{2K_1 + K_2}\,\frac{d\,(F^2)}{d\nu}, \quad . \quad . \quad . \quad (119)$$

and this must be the total electric force on the sphere.

Writing x, y, z respectively for ν in this formula we get X, Y, Z the component forces in the direction of these variables. The direction of the resultant force on the sphere is that for which $d\,(F^2)/d\nu$ is a maximum, and in which F^2 increases. The direction therefore in which the sphere tends to move is towards a place of maximum value of F^2; that is, in which the value of F is numerically greatest without distinction of sign.

(2) on Conducting Sphere.

For a conducting sphere (119) becomes

$$-\,\frac{dE}{d\nu} = r^3\,\frac{d\,(F^2)}{d\nu}, \quad . \quad . \quad . \quad . \quad . \quad (120)$$

and the sphere tends to move in the same direction as the dielectric sphere.

Force in Variable Field: (2) on Conducting Sphere.

Since, as we have seen, there is no place of maximum or minimum potential in space not occupied by any part of the electrification, a point-charge, or small sphere supposed uniformly electrified, would nowhere be in stable equilibrium except in contact with some part of the electrification; and the proposition may be extended to any electrified body. Hence in the cases here considered the spheres move along the line of greatest variation of force towards a place where the force is numerically greatest. Generally, this is the direction in which all bodies of small dimensions, placed in the electric field without charge, tend to move.

By (119) and (120) $(K_2 - K_1)/(2K_1 + K_2)$ is the ratio of the force on a dielectric sphere of specific inductive capacity K_2 to the force on a conducting sphere of the same radius placed at exactly the same place in the field of specific inductive capacity K_1.

This relation has been used by Boltzmann for the determination of specific inductive capacities (see Chap. VII.).

We shall now apply the results stated above to one or two important cases:—

Vacuum Condenser with Dielectric Layer of Sp. Ind. Cap. K.

(1) An electric field consists of two regions, one bounded by equipotential surfaces, and filled with a dielectric of specific inductive capacity K the same in all directions, and the other, the remainder of the space within the zero equipotential surface, occupied by a dielectric of unit specific inductive capacity. It is instructive to refer this example directly to the thermal

analogy. The analogue of the electrified system is a geometrically corresponding system of heat-sources and isothermal surfaces in a medium of conductivity everywhere unity, except in a region bounded by isothermal surfaces, where the conductivity is k. Suppose the whole medium at first of unit conductivity, and that then a medium of conductivity k is substituted for the former medium in the space referred to, while everything else remains unaltered. The effect of introducing the medium of (say) higher conductivity is to diminish the difference of temperature between the inner and outer surfaces of the new medium in the ratio of 1 to k, since everywhere in that medium the flux along a line of flow becomes $-kdv/dr$, which, as the generation of heat is unchanged, must be equal to the former value of $-dv/dr$. Hence also the flux at every point which is not in the new medium is unchanged, and we have therefore at every such point the same gradient of temperature as before, and therefore also the same difference of temperature as before, between any point of the system of sources and the inner surface of the new medium, and between any point in the outer surface of the new medium and the surface of zero temperature. If then the temperature of the inner surface was formerly v, and that of the outer surface v', the temperature of any point of the source has been lowered by the introduction of the medium of conductivity k by an amount $(v-v')(k-1)/k$.

Thermal Analogue.

Effect of Layer in Lowering Potential of System.

In precisely the same way in the electrical problem, if the electric charges are kept the same, the electric force at every point inside or outside the new medium

is unaltered, and, at every point within the substance of the medium itself is changed from its former value F to F/K, and the potential of any part of the electrified system is lowered by the amount $(V - V')(K-1)/K$, where V and V' are the respective potentials of the inner and outer separating surfaces of the media.

Capacity of Condenser with Dielectric of Sp. Ind. Cap. K.

If the new medium fill the whole space between the electrified system and the surface of zero potential $V' = 0$, the potential V of any part of the system has been diminished in the ratio of 1 to K, and the charge of the whole system necessary to produce a given potential at any part of it has therefore been increased in the ratio of K to 1; that is, the electrostatic capacity of the system has been increased in this ratio.

Dielectric Layer equivalent to Surface Distribution of Electricity.

The same results would be obtained by imagining the medium of inductive capacity K replaced by a medium of unit inductive capacity, and the internal and external surfaces of the region electrified so that the surface density at any point of the inner surface is

$$+\frac{K-1}{4\pi}\frac{dV}{d\nu},$$

and at any point of the outer surface

$$-\frac{K-1}{4\pi}\frac{dV}{d\nu},$$

where $dV/d\nu$ is the rate of variation outwards along a line of force passing through the point taken in the first case just inside, in the second case just outside, the

Dielectric Layer equivalent to Surface Distribution of Electricity.

region in question. These being equilibrium distributions would not alter the actual distribution, and the force inside and outside the region at any point would be the same as before, while within the region it would be diminished at any point in the ratio of 1 to K.

We see in the same way that if the specific inductive capacities, instead of being 1 and K, were respectively K_1 and K_2, the difference of potential between the two sides of the layer K_2 would be less than its value for the same space occupied by the medium K_1 in the ratio of K_1 to K_2, and the density of the imaginary distribution described in the last paragraph would be

$$+\frac{K_2 - K_1}{4\pi}\frac{dV}{d\nu}$$

for the inner surface, and for the outer surface

$$-\frac{K_2 - K_1}{4\pi}\frac{dV}{d\nu}.$$

Condenser composed of Layers of different Dielectrics

(2) The same method applies to the case of a field composed of dielectrics of inductive capacities K_1, K_2, K_3, &c., each bounded by equipotential surfaces, and arranged in this order outwards from the electrified system, which we suppose in the medium K_1. Let V be the potential of any part of the electrified system, V_1 the potential of the outer surface of K_1 and the inner surface of K_2, V_2 the potential of the outer surface of K_2 and the inner surface of K_3, and so on. Then if K_1 alone were replaced by vacuum, $V - V_1$

Condenser composed of Layers of different Dielectrics.

would become $K_1(V - V_1)$ the other differences of potential remaining the same as before; if K_2 were then replaced by vacuum, $V_1 - V_2$ would become $K_2(V_1 - V_2)$, and so on. Hence, if all the media were replaced by vacuum, the potential of any part of the electrified system would be changed from V to

$$K_1(V - V_1) + K_2(V_1 - V_2) + \&c.$$

Hence, if C be the new value of the electrostatic capacity of the system and C' its former value, we have

$$\frac{C}{C'} = \frac{V}{K_1(V - V_1) + K_2(V_1 - V_2) + \&c.} \quad (121)$$

Maxwell's Theory of Stress in the Dielectric Medium.

Maxwell * has considered a dielectric medium surrounding an electrified system as in a state of strain under stresses consisting of a tension (as in a stretched wire or cord) acting at each point along the direction of the electric force, and an equal pressure at the same point in all directions at right angles to that of the electric force. The amount of the tension and pressure (each taken in units of force per unit of area) at any point at which the electric force is F in a medium of specific inductive capacity K is $KF^2/8\pi$; that is, equals (p. 34 above) the electric energy of the medium per unit of volume at that point.

"Electric Displacement."

Further, he has regarded the electric charge of the system as the surface manifestation of a change which took place in the medium when the electrification was set up. This change he has called *Electric Displacement*,

* *El. and Mag.*, vol. i., sec. ed., pp. 59—67 and 153—156.

and consists in a passage, across every surface drawn in the medium so as to enclose the electrified system, of a quantity of electricity equal to the charge on the system, so that the introduction of a charged system within a closed space does not produce any change in the total quantity of electricity within the space. Thus when one coating of a condenser is charged positively an equal quantity of positive electricity passes towards the other coating across every intermediate surface, and the charges on the coatings are to be regarded as the charges of the surfaces of the separating dielectric. If any change take place in the charge, a corresponding change takes place in the displacement. Hence when a quantity of electricity is transferred from one coating, A, to the other, B, as when charge or discharge takes place along a wire connecting them, an equal quantity of electricity crosses every section of the dielectric from B towards A. If therefore we regard the process of displacement as an electric current, the dielectric and the wire constitute a closed circuit round which a current passes so long as any change in the electric state of the system is taking place.

Electric Displacement.

Surface-Integral of Electric Induction.

The magnitude of the electric displacement is $KF/4\pi$, and the displacement across any element δs of a surface drawn everywhere at right angles to the lines of induction is $KF\delta s/4\pi$. The integral of this expression taken over the surface is the whole quantity of electricity in the form of a charge within the surface.

The ratio $4\pi/K$ of the electric force to the electric displacement Maxwell has called by analogy the *Co-efficient of Electric Elasticity* of the medium. In

Electric Elasticity.

virtue of the electric elasticity a force opposing the displacement is set up which restores the medium to its former state when the electric force is removed. In a conducting wire this elastic force is continually giving way, and being restored by the displacement continually going on, which therefore constitutes an electric current.

CHAPTER II.

THEORY OF FLOW OF ELECTRICITY.

SECTION I.

GENERAL CONSIDERATIONS. STEADY FLOW.

HITHERTO we have been dealing with the statical phenomena of electricity: it is necessary now, before entering on the subject of even purely statical measurements, to briefly consider some of the phenomena and laws of current electricity. We shall not, however, here deal with any part of the great subject of electromagnetism, but reserve that for a special chapter preliminary to an account of electromagnetic measurements.

Phenomena attending Change of Equilibrium state of Electric System.

When two conductors are brought into contact either directly, or by means of an interposed conductor in contact with both, there ensues in all cases in which the two conductors are of the same material (if not in all cases whatever) an equalization of their potentials. This equalization does not take place instantaneously, although for most practical purposes the time during which the change takes place may be regarded as infinitely short.

Diminution of Electric Energy of System.

In every such case there is a diminution of the electric energy of the conductor which is at the higher potential accompanied by a fall of its potential, an

increase of the electric energy of the other conductor with a rise of its potential, and a diminution of the electric energy of the whole system. To estimate these changes in a specific case we shall suppose that the conductors are brought, without any change of position, into contact by means of a thin conducting wire such that its capacity may be neglected in comparison with that of either of the conductors connected. After contact therefore these may be regarded as one conductor with charge equal to the sum of the separate charges before contact, and capacity equal to the sum of the separate capacities. Let then Q_1, Q_2, be the charges of the conductors before contact, K_1, K_2, their capacities.

Diminution of Electric Energy of System.

The energy before contact was $\frac{1}{2}Q_1^2/K_1 + \frac{1}{2}Q_2^2/K_2$; after contact it is $\frac{1}{2}(Q_1+Q_2)^2/(K_1+K_2)$. The diminution of energy is therefore given by the expression

$$\frac{1}{2}\left\{\frac{Q_1^2}{K_1}+\frac{Q_2^2}{K_2}-\frac{(Q_1+Q_2)^2}{K_1+K_2}\right\}=\frac{1}{2}\frac{(Q_1K_1-Q_2K_2)^2}{K_1K_2(K_1+K_2)}\ldots (1)$$

which is essentially positive.

The energy represented by this expression is transformed into heat which, when no magnetic or chemical work is done, takes the form of heat given out partly in the intermediate conductor, partly in the conductors themselves, and partly in a spark when the contact is made.

Equivalent of Energy Lost.

The passage to the new state of equilibrium may be made to occupy a longer or a shorter time according to the arrangement adopted. For example, if the conductors be the opposite plates of a condenser, and the joining conductor be a long thin wire wound

Duration of Transition to new state.

into a helix containing an iron core, the time taken to annul a given difference of potential between the conductors may be made so long as to be capable of exact measurement.

Electric Current.

During the time of transition there is a flow of electricity from one conductor to another, and this is what is called an *Electric Current.*

Measure of Strength of Current.

The average *Strength of Current* over any cross-section of the conducting wire is measured by the limit towards which the ratio of the quantity of electricity, which passes the cross-section in a small interval of time, to the magnitude of the interval approaches as the interval is made smaller and smaller; that is, it is the time-rate of flow of electricity across the section. Hence, in all cases in which the current may be regarded as having the same value at any one instant over every cross-section, the time-rate at which one conductor loses and at which the other gains charge is equal to the current. We shall see later how current-strengths may be measured experimentally.

Definition of Steady Current.

The current has the same value at every cross-section when the capacity of the connecting conductor is negligible in comparison with that of each of the conductors connected, and also when the current is *steady;* that is, when its value for any one cross-section does not vary with the time; but in many cases of currents of very short duration the assumption of the fulfilment of this condition must be regarded as giving results which are only approximately true, and in other cases, for example that of a submarine cable, cannot be made at all.

When, however, this condition is fulfilled, we see that an electric current may be compared to a current of an incompressible fluid between two vessels communicating by a rigid canal, which opens only into the vessels and is kept full by the current. The difference of potentials between the conductors is analogous to the difference of pressures between the two vessels, and the current across any section of the conductor to the time-rate of flow of the fluid across any section of the channel. Since the fluid is incompressible and the channel is kept full and unaltered in dimensions, the time-rate of flow, however it may vary with the time, will have at any one instant the same value at every cross-section. Hydrokinetic Analogy.

The time-rate of loss of energy at any instant is plainly equal to the product of the current and the difference of potentials between the conductors. Denoting the potentials of the conductors by V_1 and V_2, and the current or time-rate of loss of charge by γ, and using A to denote time-rate of working, or *Activity*, we have for this case Time-rate of Loss of Energy.

$$A = (V_1 - V_2)\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This expression is of course precisely similar to that which in the hydrokinetic analogy expresses the rate of working of the current of fluid.

The flow of electricity in bodies is also exactly analogous to the conduction of heat and to the diffusion of liquids and gases, and the mathematical theory common to these two classes of phenomena may be used also to give results in the electrical problem. We shall see below that the amount of flow depends on the nature of the substance exactly as the flow of heat depends on Analogy of Electric to Thermal Conduction.

Analogy of Electric to Thermal Conduction.

the thermal conductivity of the substance. In fact, if we take a difference of temperature as the analogue of a difference of potentials, rate of flow of heat across an area as the analogue of an electric current, and conductivity of a substance for heat (taken as independent of temperature according to Fourier's supposition) as the analogue of a quantity which we call the *Specific Electric Conductivity* of a substance, we may transfer the equations of heat conduction bodily to the theory of flow of electricity. For example, the theory given in Section V., Chapter I., for electrostatic induction in different media, can be at once translated into a theory of electric flow, or, as it is also called, conduction of electricity, in different media; and we shall see below that the results of that section are available without modification.

The analogies we have referred to are only some of those which exist between the mathematical theories of electricity and magnetism, the motion of fluids (including diffusion), and the conduction of heat; and it seems highly probable that some of these analogies are consequences of hitherto undiscovered mutual relations of the phenomena.

Electric Resistance of a Linear Conductor.

It is found experimentally by measuring with a delicate electrometer, that between any two cross-sections A and B of a homogeneous wire, which is not in motion in a magnetic field, and along which a steady current of electricity is kept flowing by any means, there exists a difference of potentials, and that if the wire be of uniform section throughout, the difference of potentials is in direct proportion to the length of wire between the cross-sections. It is found, further, that if

Electric Resistance of a Linear Conductor.

the difference of potentials between A and B is kept constant, and the length of wire between them is altered, the strength of the current varies inversely as the length of the wire. Again, if the length of wire and the difference of potentials between A and B be kept the same while the cross-sectional area of the wire is increased or diminished, the current is increased or diminished in the same ratio. Hence the wire is said to oppose to the current a *resistance* which is directly proportional to the length of wire between the two cross-sections, and inversely proportional to the cross-sectional area of the wire.

If for any particular wire measurements of the current strength in it be made for various measured differences of potentials between two cross-sections, the current strengths are found to depend only on, and to be in simple proportion to, the differences of potential so long as there is no sensible heating of the wire.

Ohm's Law.

If γ be put for the strength of the current flowing in a wire of resistance R between two cross-sections at potentials V_1, V_2 respectively, these results are all expressed, and unit resistance is defined, by the equation

$$\gamma = \frac{V_1 - V_2}{R}. \qquad \qquad (3)$$

Electromotive Force defined.

This is equivalent to a relation given by G. S. Ohm,* and is hence called Ohm's Law. Ohm used the expression "Gefälle der Elektricität" for a quantity which, in the earlier works which appeared after the publication of his essay, was called "Difference of Tensions," but

* *Die Galvanische Kette mathematisch bearbeitet*, Berlin, 1827.

Electromotive Force defined.

which is now recognized as proportional to $V_1 - V_2$; and it is still usual to give a special name to difference of potentials when considered in connection with the flow of electricity. Thus the name *Electromotive Force* is frequently given to the difference of potentials between two points or two equipotential surfaces in a homogeneous conductor, when thus considered with reference to flow of electricity from one to the other, and in accordance with custom and authority the term may be thus employed. A somewhat more general sense in which the term is used will presently be explained. It is to be carefully remembered, however, that electromotive force is not a *force:* the two words must be taken *together* as a single term having the meaning assigned to it in its definition.

A constant difference of potentials may be maintained between the extremities of a homogeneous conductor, and therefore also a current maintained in the conductor, in several different ways: for example, by a voltaic battery, a thermo-electric pile, or a dynamo-electric or magneto-electric machine. Particulars regarding different forms of voltaic batteries, and the practical construction of other electric generators, are given in various treatises; at present we deal only with principles which are generally applicable, reserving for consideration later their applications in particular cases.

Electromotive Force in a Conductor.

Equation (3) is not fulfilled in general by a conductor made up of different homogeneous portions, put end to end, or by a conductor moving across the lines of force of a magnetic field. For such cases

$$\gamma = \frac{V_1 - V_2}{R} + \frac{E}{R} \quad . \quad . \quad . \quad . \quad . \quad (3a)$$

Electromotive Force in a Conductor.

where V_1, V_2 denote as before the potentials at the extremities of the conductor, and R the sum of the resistances of the homogeneous portions of the conductor in the former case, or the actual resistance of the conductor in the latter. The conductor in such cases is said to *contain*, or to be the *seat of, an electromotive force E*, or (as frequently in what follows) an electromotive force E is said to be *in* the conductor. The total electromotive force producing a current in the conductor is now $V_1 - V_2 + E$. Since in a heterogeneous conductor (3) applies in the first case to every part, except any, however small, which includes a surface of discontinuity, the electromotive force is said to have its seat at the surface or surfaces of discontinuity. In the other case electromotive force has its seat in every part of the conductor moving in the field, according to a law which we shall afterwards discuss.

Application of Ohm's Law to Heterogeneous Circuit.

In a circuit composed of different homogeneous conductors let adjacent points be taken on opposite sides of each surface of continuity, and let the difference of potentials between the pair of points in each conductor be measured; the sum of these differences taken in order round the circuit is equal to the sum of the parts of E contributed by the discontinuities. For going round in the direction of the current from a point (not in a surface of discontinuity) to the same point again we have $V_1 = V_2$ and

$$\gamma = \frac{E}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3b)$$

Total Electromotive Force of a Circuit.

But denoting the successive homogeneous conductors in their order round the circuit by the suffixes 1, 2, ... n, and the differences between their extremities by

$$V_1 - V'_1,\ V_2 - V'_2, \ldots .\ V_n - V'_n,$$

and the corresponding resistances by $R_1, R_2, \ldots . R_n$, we have

$$\gamma = \frac{V_1 - V'_1}{R_1} = \frac{V_2 - V'_2}{R_2} = \ldots \frac{V_n - V'_n}{R_n} = \frac{\Sigma(V - V')}{R} \quad (3c)$$

Hence

$$\Sigma(V - V') = E. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

E is called the *total* electromotive force in the circuit, or simply the electromotive force of the circuit.

In Chapter VIII. will be found an account of experimental methods used for the verification of Ohm's Law, and details as to its application to chains of conductors of different substances. We will consider here as an example of the principles just stated its application to the case of a simple voltaic cell composed of two plates of dissimilar metals connected by a liquid, for example, copper and zinc immersed in hydrochloric acid, and connected externally by a copper wire.

Case of Simple Voltaic Cell.

Let c and z denote the copper and zinc plates, l the liquid between them. By the theory of the voltaic cell now generally adopted, there is a certain finite difference of potential on the two sides of the junction of the dissimilar metals, and on the two sides of each junction of a metal with the liquid. We may suppose for simplicity the plates to be such that they add no sensible resistance to the circuit, and that therefore the potential may be taken as the same at every point

Case of Simple Voltaic Cell.

of each. Let V_a denote the potential of the copper plate; V_b the potential of the copper wire close to its junction with the zinc plate; V_{lz} the potential of the stratum of the liquid close to the zinc plate; and V_{lc} the potential of the stratum of the liquid close to the copper plate. The difference of potentials between two points in the copper conductor near its ends is therefore $V_a - V_b$, and that between the two isdes of the liquid is $V_{lz} - V_{lc}$. Both of these differences are positive, and the current flows from the copper plate to the zinc plate through the wire, and from the zinc plate to the copper through the liquid. Further it is an experimental fact, as we shall see later, that the current across any cross-section is the same at every part of the circuit. Calling R the resistance of the copper conductor joining the plates, and r the resistance of the liquid of the cell, we have by (3)

$$\gamma = \frac{V_a - V_b}{R} = \frac{V_{lz} - V_{lc}}{r},$$

and therefore also

$$\gamma = \frac{V_a - V_{lc} + V_{lz} - V_b}{R + r}.$$

Electromotive Force of Voltaic Cell

But $V_a - V_{lc}$ is the finite difference between the potential of the copper plate and that of the liquid in contact with it, and $V_{lc} - V_b$ is similarly the difference between the potential of the liquid in contact with the zinc plate and that of the extremity of the copper wire adjacent to the zinc plate, and the sum of these two

differences constitutes what is called the *Electromotive Force of the cell.* Calling this E, we have

$$\gamma = \frac{E}{R + r}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

Any other case, however more complicated, might be treated in a similar manner.

Distribution of Potential in Circuit. If V be the difference of potentials between any two points in the copper wire, R the resistance of the wire between these two points, and r the remainder of the resistance in circuit, we have from the equations

$$\gamma = \frac{V}{R} = \frac{E}{R + r},$$

the result

$$V = E\frac{R}{R + r}. \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

Activity in Circuit. The activity in the wire is by (2)

$$A = V\gamma = \frac{V^2}{R} \quad . \quad . \quad . \quad . \quad . \quad . \qquad (7)$$

and for the whole circuit

$$A = E\gamma = \frac{E^2}{R} \quad . \quad . \quad . \quad . \quad . \qquad (7\ bis)$$

By (7) the activity in any wire not containing an electromotive force can always be found, whatever be the arrangement of which it forms part. The activities in the different parts of more complicated circuits containing electromotive forces of different kinds will be considered in the chapter on the Measurement of Electric Energy.

Electromotive Force and Current of Voltaic Battery:

(1) Arrangement in Series;

If instead of a single cell we have a battery of several cells, its electromotive force is found in exactly the same manner by summing all the finite differences of potential at the surfaces of separation of dissimilar substances in the circuit. Hence if there be n cells in the battery joined in series, that is to say the zinc plate of the first cell joined to the copper plate of the second cell, the zinc plate of the second to the copper plate of the third, and so on to the last cell, the total electromotive force of the arrangement, if the cells have each the electromotive force E, is nE. If the copper plate of the first cell and the zinc plate of the last be joined by a wire, and R denote as before its resistance, r the internal resistance of each cell, a current of strength γ given by the equation

$$\gamma = \frac{nE}{R + nr} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

will flow in the wire. This equation may be written

$$\gamma = \frac{E}{r + \frac{R}{n}},$$

which shows that when n is so great that R/n is small in comparison with r, little is added to the value of γ by further increasing the number of cells in the battery.

(2) Arrangement in Multiple Arc.

The method of joining single cells in series is advantageous when R is large, but when R is comparatively small it fails as shown above, and it is necessary then to diminish r as much as possible. The value of r is, for cells in which, as is generally the case, each plate nearly

Arrangement in Multiple Arc.

covers the cross-section of the liquid, nearly in the inverse ratio of the area of the plates, and directly as the distance between them. Hence, by increasing the area of the plates and placing them as close together as possible, the resistance may be diminished. One large cell of small resistance may be formed of several small cells by putting all the copper plates into metallic connection with one another, and similarly all the zinc plates. Several compound cells of large surface thus made may be joined in series. The electromotive force of each compound cell will be E as in a simple cell, but if m cells be joined so as to form one compound cell its resistance will be r/m. If n of these compound cells be joined in series, we have, calling the total external resistance R,

$$\gamma = \frac{n\,E}{R + n\dfrac{r}{m}} = \frac{m\,n\,E}{m\,R + n\,r} \quad . \quad . \quad . \quad . \qquad (9)$$

Condition of Maximum Current through given Resistance

If R be not too great, and we have a proper number of cells, it is possible to arrange the battery so that γ may have a maximum value. There being $m\,n$ cells in the battery the numerator of the above value of γ does not change when the arrangement of cells is varied, and therefore, in order that γ may have its greatest possible value, $m\,R + n\,r$ must be made as small as possible. But identically,

$$m\,R + n\,r = (\sqrt{m\,R} - \sqrt{n\,r})^2 + 2\sqrt{m\,n\,R\,r}.$$

As the last term on the right-hand side does not vary with the arrangement of the battery, it is plain that

$m R + n r$ will have its smallest value when $\sqrt{mR} - \sqrt{nr}$ vanishes, that is when $m R = n r$ or $R = nr/m$, or, in words, when the total external resistance of the circuit is equal to the internal resistance of the battery. It may not be possible in practice so to join a given battery as to fulfil this condition, but if the strongest possible current is required it should be fulfilled as nearly as possible. This method of arranging the battery is called joining it in *multiple arc*.

Condition for Maximum Current not that for Maximum Efficiency.

It is to be carefully observed that this theorem applies only to the case in which we have a given battery and have to arrange it so as to produce the *greatest current* through a given external resistance R; and the fallacy is to be avoided of supposing that of two batteries of equal electromotive force, but one having a high, the other a low, resistance, the former is better adapted for working through a high external resistance. Nor is this method of using the battery to be confounded with the most *economical* method. By this arrangement the greatest rate of working in the external part of the circuit is obtained; for by (7 *bis*) the total rate of working is $nE\gamma$, and the part of this which belongs to the external conductor is $mnE\gamma R/(mR + nr)$, which is a maximum under the same conditions as γ. As much energy is thus given out in the battery itself as in the external resistance, and it is plain that for economy as little as possible of the energy of the battery must be spent in the battery itself, and as much as possible in the working part of the circuit. Hence for economical working the internal resistance of the battery and the resistance of the wires connecting the battery with the

working part of the circuit must be made as small as possible. We shall return to this question in a later chapter.

Theory of System of Linear Conductors.

We shall now consider shortly the theory of a system of linear conductors (homogeneous wires of uniform section) in which steady currents are flowing.

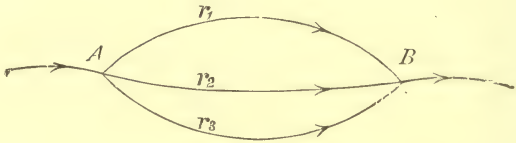

FIG. 33.

Principle of Continuity.

It has been stated above that when a steady current is kept flowing across any cross-section of a conductor, the current strength is the same across every other section of the conductor; or, in other words, that the flow of electricity at any instant into any portion of the conductor is equal to the flow out of the same portion. This is what is called the *principle of continuity* as applied to the case of a *steady* flow of electricity. By the same principle we have, in the case in which steady currents are maintained in the various parts of a network of conductors, the theorem that the total flow of electricity towards the point at which several wires meet is equal to the total flow from that point. Thus the current arriving at A (Fig. 33) by the main conductor is equal to the sum of the currents which flow from A by the arcs which connect it with B.

Equivalent Resistance of a Multiple Arc.

By Ohm's law, if two points A and B, between which a difference of potentials V is maintained, be connected by two wires of resistances r_1 and r_2, the current in that of resistance r_1 will be V/r_1 and in the other V/r_2. But if γ be the whole current flowing in the circuit we have by the principle of continuity

$$\gamma = \frac{V}{r_1} + \frac{V}{r_2} = \frac{V}{R},$$

where R is the resistance of a wire which might be substituted for the double arc between A and B without altering the current in the circuit. Hence,

$$\left.\begin{aligned} \frac{1}{r_1} + \frac{1}{r_2} &= \frac{1}{R} \\ \text{and} \qquad R &= \frac{r_1 r_2}{r_1 + r_2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Definition of Conductivity.

The reciprocal of the resistance R of a wire, that is, $1/R$, is called its *conductivity*. Equation (9) therefore affirms that the conductivity of a wire, the substitution of which for r_1 and r_2 between A and B, would not affect the current in the circuit, is equal to the sum of the conductivities of the wires r_1 and r_2. From equation (10) we see that the resistance R of this equivalent wire is equal to the product of the resistances of the two wires divided by their sum.

If for r_2 we were to substitute two wires having an equivalent resistance, so that A and B should be connected, as in Fig. 33, by three separate wires of resistances

r_1, r_2, r_3, we should have in the same manner for the current in r_1, V/r_1; in r_2, V/r_2; in r_3, V/r_3, and

$$\left.\begin{aligned} \frac{1}{R} &= \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \\ R &= \frac{r_1\, r_2\, r_3}{r_1\, r_2 + r_2\, r_3 + r_3\, r_4} \end{aligned}\right\} \quad . \ . \ . \ (11)$$

Conductivity and Resistance of Multiple Arc.

Generally, if two points A and B are connected by a multiple arc consisting of n separate wires, the conductivity of the wire equivalent to the multiple arc connection is equal to the sum of the conductivities of the n connecting wires; and its resistance is equal to the product of the n resistances divided by the sum of all the *different* products which can be formed from the n resistances by taking them $n - 1$ at a time.

As a simple example, we may take the case of a number n of incandescence lamps joined in multiple arc. If the resistance of each lamp and its connections be r, the equivalent resistance between the main conductors, the resistance due to the latter being neglected, is by (11) $r^n/n\, r^{n-1} = r/n$. Thus if r be 60 ohms when the lamp is incandescent, and there be twenty lamps, their resistance to the current will be 3 ohms.

Total Electromotive Force in any Circuit in Network of Conductors.

By the considerations stated above, we at once deduce from Ohm's law the following important theorem.* In any closed circuit of conductors forming part of any linear system, the sum of the products obtained by multiplying the current in each part, taken in order round

* This theorem and the application of the principle of continuity were first stated explicitly by Kirchhoff, *Pogg. Ann.* Bd. 72, 1847, also *Ges. Abhand.*, p. 22.

the circuit by its resistance, is equal to the sum of the electromotive forces in the circuit. This follows at once by an application of Ohm's law to each part of the circuit, exactly as in the investigation in p. 145 above of the electromotive force of the circuit composed of a cell and a single conductor.

As an example of a circuit containing no electromotive forces, consider the circuit formed by the two wires (Fig. 33) of resistances r_1, r_2 joining AB. We have, for the current flowing from A to B through r_1, the value V/r_1; the product of this by r_1 is V; for the current flowing from B to A through r_2 we have $-V/r_2$, and the product of this by r_2 is $-V$: the sum is $V-V$ or zero. As another example, consider the Examples.

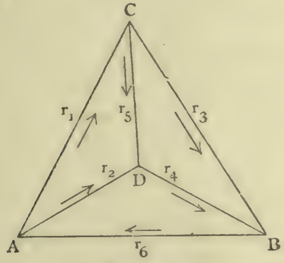

FIG. 34.

diagram, Fig. 34, of resistances r_1, r_2, r_3, r_4, r_5, between the two points A and B. By what we have seen, if V_a, V_b, V_c, V_d, be the potentials at A, B, C, D, respectively, the current from A to C is $(V_a - V_c)/r_1$, from C to D $(V_c - V_d)/r_5$, and from D to A $(V_d - V_a)/r_2$. Hence, taking the sum of the products of these current

strengths by the corresponding resistances for the circuit *ACDA*, we get

$$V_a - V_c + V_c - V_d + V_d - V_a = 0 \quad . \; . \; . \; (12)$$

Problem of Two Points joined by Network of Five Conductors.

To illustrate the use of the principles which have been established, we may apply them to find the current strength in r_5 (Fig. 34) when r_6 contains a battery of electromotive force E. Let r_6 be the resistance of the battery and the wires connecting it with A and B, and let γ_1, γ_2, &c., be the strengths of the currents flowing in the resistances r_1, r_2, &c., respectively, in the directions indicated by the arrows. By the principle of continuity we get

$$\left.\begin{aligned} \gamma_3 &= \gamma_1 - \gamma_5 \\ \gamma_4 &= \gamma_2 + \gamma_5 \\ \gamma_6 &= \gamma_1 + \gamma_2 \end{aligned}\right\} \quad . \; . \; . \; . \; . \; (13)$$

Applying the second principle to the circuits *BACB*, *ACDA*, *CBDC*, and using equation (12), we obtain the three equations,

$$\left.\begin{aligned} &\gamma_1 (r_1 + r_3 + r_6) + \gamma_2 r_6 - \gamma_5 r_3 &&= E \\ &\gamma_1 r_1 - \gamma_2 r_2 + \gamma_5 r_5 &&= 0 \\ &\gamma_1 r_3 - \gamma_2 r_4 - \gamma_5 (r_3 + r_4 + r_5) &&= 0 \end{aligned}\right\} \; (14)$$

Eliminating γ_1 and γ_2, we find

$$\gamma_5 = \frac{E\,(r_2 r_3 - r_1 r_4)}{D} \quad . \; . \; . \; . \; (15)$$

where

$$D = r_5 r_6 (r_1 + r_2 + r_3 + r_4) + r_5 (r_1 + r_3)(r_2 + r_4) + r_6 (r_1 + r_2)(r_3 + r_4) + r_1 r_3 (r_2 + r_4) + r_2 r_4 (r_1 + r_3) \quad . \; . \; (16)$$

By substituting for γ_2 in the second and third of equations (14) its value $\gamma_6 - \gamma_1$, we get,

Problem of Two Points joined by Network of Five Conductors.

$$\left.\begin{aligned} \gamma_1 (r_1 + r_2) + \gamma_5 r_5 \qquad\qquad\qquad\quad - \gamma_6 r_2 &= 0 \\ \gamma_1 (r_3 + r_4) - \gamma_5 (r_3 + r_4 + r_5) - \gamma_6 r_4 &= 0 \end{aligned}\right\} . \quad (17)$$

From these we obtain by eliminating γ_1,

$$\gamma_5 = \frac{\gamma_6 (r_2 r_3 - r_1 r_4)}{r_5 (r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)} \quad . \quad (18)$$

By means of equations (15) and (18) we can very easily solve the problem of finding the equivalent resistance of the system of five resistances r_1, r_2, &c., between A and B. For let R be this equivalent resistance, since γ_6 is the current flowing through the battery, we have $\gamma_6 = E/(r_6 + R)$. Substituting this value of γ_6 in (18), equating the values of γ_5 given by (15) and (18), and solving for R, we get

$$R = \frac{r_5 (r_1 + r_3)(r_2 + r_4) + r_1 r_3 (r_2 + r_4) + r_2 r_4 (r_1 + r_3)}{r_5 (r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)} \quad (19)$$

Principle of Superposition.

It follows from Ohm's law, and the theorems which have been deduced from it, that any two states of a system of conductors may be superimposed; that is, the resulting potential at any point is the sum of the potentials at the point, the current in any conductor the sum of the currents in the conductor, and the electromotive force in any circuit the sum of the electromotive forces in the circuit, in the two states of the system.

The following result, which is a direct inference from

the foregoing principles, and can be verified by experiment, will be of use in what immediately follows.

Result of Foregoing Principles.

Any two points in a linear circuit which are at different potentials may be joined by a wire without altering in any way the state of the system, provided the wire contains an electromotive force equal and opposite to the difference of potential between the two points. For the wire before being joined will in consequence of the electromotive force have the same difference of potentials between its extremities as there is between the two points, and if the end of the wire which is at the lower potential be joined to the point of lower potential it will have the potential of that point, and no change will take place in the system. The other end will then be at the potential of the other point, and may be supposed coincident with that point, without change in the state of the system. The new system thus obtained plainly satisfies the principle of continuity, and the theorem of p. 152 above, and is therefore possible; and it can be proved that it is the only possible arrangement under the condition that the state of the original system shall remain unaltered.

As a particular case of this result any two points in a linear circuit which are at the same potential may be connected, either directly or by a wire of any resistance, without altering the state of the system.

Further, it follows that if an electromotive force in one conductor, *A*, of a linear system can produce no current in another conductor, *B*, of the system, either conductor may be removed without altering the current in the other. For let the conductor, *A*, be removed: the

Result of Foregoing Principles.

potentials at the points of the system at which it was attached will in general then be altered. And since any two points in a linear system between which there is a difference of potentials may, without altering the state of the system in any way, be joined by a wire which contains an electromotive force equal and opposite to the difference of potentials, we may suppose the conductor replaced with an electromotive force in it equal to the difference of potentials now existing between the two points, and its presence or removal will not affect the current in any part of the system. But the same result may be attained, of course, without removing the conductor, by simply placing within it the required electromotive force, and this by hypothesis does not affect the current in the other conductor. Hence the removal of the conductor, A, does not affect the current in the other. Again, by the first reciprocal relation below (p. 159), if an electromotive force in A can produce no current in B, an electromotive force in B can produce no current in A. Hence the same proof shows that B may be removed without affecting the current in A.

Theorems regarding a Network of Conductors.

If A, B, C, D be four points of meeting in a network of linear conductors, in one wire of which joining AB there is an electromotive force, while CD is connected by one or more separate wires, the network can be reduced to a system of six conductors arranged as in Fig. 34, and such that the wires AB, CD, the currents in them, and the potentials at their extremities remain unchanged. For currents will enter any one mesh of the network at certain points and leave it at certain

Theorems regarding a Network of Conductors.

other points. One of the former points must be the point of maximum potential in the mesh, one of the latter the point of minimum potential. The circuit of the mesh, therefore, consists of two parts joining these two points, and to any point in one of the parts will correspond a point of the same potential in the other part. We may therefore suppose every point in one in coincidence with points of the same potential in the other; that is, the mesh replaced by a single wire joining the two points, and such that the currents entering or leaving it by wires joining it to the rest of the system, and the potentials at the points of junction, are not altered.

Since the only electromotive force is in the wire AB, the current must enter the network at one of its extremities, A, say, and leave at the other extremity, B. A and B are therefore the points of maximum and minimum potential of the network. Hence we can replace the meshes of the system one by one by single wires, keeping CD unaltered until we have reduced the network to two meshes, one on each side of CD, connected by single wires to A and B respectively. Each mesh and connecting-wire can be replaced by two wires joining A and B respectively with CD, and thus the whole system is reduced to an equivalent system of the form shown in Fig. 34. We can now deduce from this simple system relations for the currents and potentials in the conductors AB, CD, which will hold for these conductors in the more complex system.

Let the electromotive force hitherto supposed acting in AB be transferred to CD, while the resistances

r_5, r_6 are maintained unaltered. The value of γ_6 will be obtained from (15) by retaining the numerator unaltered and interchanging r_5 and r_6, $r_1 + r_2$ and $r_1 + r_3$, $r_3 + r_4$ and $r_2 + r_4$ in D. But these interchanges will not effect any alteration in the value of D, and hence the new value of γ_6 is equal to the former value of γ_5. Hence the theorem:—An electromotive force which, placed in any conductor C_l of a linear system, causes a current to flow in any other C_p would, if placed in C_p, cause an equal current to flow in C_l.

Theorems regarding a Network of Conductors.

First Reciprocal Relation.

If the arrangement is such that when the electromotive force is in C_l the current in C_m is zero, the current in C_l will be zero when the electromotive force is in C_m; and no electromotive force in one will produce a current in the other. The two conductors are in this case said to be *conjugate.*

Conjugate Conductors.

We can easily obtain another important theorem. The five conductors AC, AD, BC, BD, CD, in Fig. 34 may be regarded as the reduced equivalent of a network of conductors, at one point of which, A, a current of amount γ_6 enters, and at another point of which, B, the same current leaves. Multiplying the expression for γ_5 by r_5, we get for the difference of potentials between C and D the value

$$\gamma_5 r_5 = \frac{\gamma_6 r_5(r_2 r_3 - r_1 r_4)}{r_5(r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)}. \quad (20)$$

But the resistance of the system of five conductors, between the points CD, is

$$\frac{r_5(r_1 + r_2)(r_3 + r_4)}{r_5(r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)},$$

Conjugate Conductors.

and if a current of amount γ_6 enter at C and leave at D, the difference of potentials between C and D will be equal to this expression multiplied by γ_6. The product multiplied by $r_1/(r_1 + r_2)$ is the difference of potentials between C and A, and multiplied by $r_3/(r_3 + r_4)$ is the difference of potentials between C and B. Hence the difference of potentials between A and B is the difference of these products, or

$$\frac{\gamma_6 r_5(r_1 r_4 - r_2 r_3)}{r_5(r_1 + r_2 + r_3 + r_4) + (r_1 + r_2)(r_3 + r_4)},$$

the same value as that given in (20) for the difference of potentials between C and D. Hence the theorem :—

Second Reciprocal Relation.

If to a current entering at one point A of a linear system and leaving at another point B, there correspond a certain difference of potentials between two other points C and D, then to an equal current entering the system at C and leaving at D there will correspond the same difference of potentials between A and B.*

Effect of Addition of a Single Wire to Linear System.

The following result is easily proved, and is frequently useful. If the potentials at two points A, B, of a linear system of conductors containing any electromotive forces, be V, V' respectively, and R be the equivalent resistance of the system between these two points, then if a wire of resistance, r, be added, joining AB, the current in the wire will be $(V - V')/(R + r)$. In other words the linear system, so far as the production of a current in

* The theorems just proved have been obtained in different ways by Kirchhoff (*Pogg. Ann.* Bd. 72, 1847, and *Ges. Abhand.* p. 22), and Maxwell (*El. and Mag.* vol. i., p. 371) from a consideration of the general theory of a linear system.

Effect of Addition of a Single Wire to Linear System.

the added wire is concerned, may be regarded as a single conductor of resistance R connecting the points AB and containing an electromotive force of amount $V - V'$. For let the points A and B be connected by a wire of resistance r, containing an electromotive force of amount $V - V'$ opposed to the difference of potentials between A and B, no current will be produced in the wire, and no change will take place in the system of conductors. Now imagine another state of this latter system of conductors in which an equal and opposite electromotive force acts in the wire between A and B, and there is no electromotive force in any other part of the system. A current of amount $(V - V')/(R + r)$ will flow in the wire. Now let this state be superimposed on the former state, the two electromotive forces in the wire will annul one another, and the current will be unchanged. The potentials at different points, and the currents in different parts, of the system, will be the sum of the corresponding potentials and currents in the two states, and will therefore, in general, differ from those which existed before the addition of the wire.

Steady Flow in Non-Linear Conductors.

So far we have considered only cases of steady flow in conductors which are called linear—that is, conductors for which it is convenient to consider the total current flowing from one equipotential surface to another, and when no electromotive force has its seat in this position of the conductor, to take the ratio of the difference of the potentials of the surfaces to this total current as the resistance between the surfaces. It is of importance, however, for the comparison of experiment with theory, to consider the distribution of the flow throughout conductors, and the forms of the equipotential surfaces in different cases, and for this purpose it is necessary to find the differential equation of the potential for the case of steady flow in any one medium, and from one medium to another. We shall consider only isotropic

media. The theory given above (pp. 103–105) for the flow of heat is directly applicable.

Ohm's Law; Non-Linear Conductors.

Assuming what is the fundamental principle of the theory of Ohm, that the rate of flow of electricity at any point, x, y, z, in any direction is directly proportional to the gradient of potential at that point and in that direction, we have for the flow of electricity per unit of time per unit of area in each of three mutually rectangular directions in an isotropic medium the values

$$-k\frac{dV}{dx}, \quad -k\frac{dV}{dy}, \quad -k\frac{dV}{dz};$$

since the flow takes place in the direction in which V diminishes. The coefficient k is the *Specific Conductivity* of the material, and is measured by the reciprocal of the resistance between two opposite faces of a centimetre cube of the substance.

Considering now an elementary rectangular parallelepiped having edges of lengths dx, dy, dz, and its centre at x, y, z, and containing within it no electromotive force, we get for the flow in the direction of x into the element the value

$$\left(-k\frac{dV}{dx} - \tfrac{1}{2}k\frac{d^2V}{dx^2}dx\right)dy\,dz,$$

and for the flow out of the element across the opposite face

$$\left(-k\frac{dV}{dx} + \tfrac{1}{2}k\frac{d^2V}{dx^2}dx\right)dy\,dz.$$

Proceeding in the same way for the other two pairs of faces, we get for the total excess of inflow above outflow

$$-k\left(\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}\right)dx\,dy\,dz.$$

If the flow is steady this must be zero, and we have

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz} = 0 \quad . \quad . \quad . \quad . \quad . \quad (21)$$

that is, Laplace's equation holds for this case. The electric density is therefore zero in the interior of such a conductor.

Surface Characteristic Equation.

At any point of a surface which separates a medium of conductivity k_1 from one of conductivity k_2, we have the equation

$$k_1\frac{dV}{d\nu_1} + k_2\frac{dV}{d\nu_2} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

where $dV/d\nu_1$, $dV/d\nu_2$, are the rates of variation outwards from the surface along normals ν_1, ν_2 drawn from the point into the media.

Putting $k_2 = 0$, we get for the equation at the surface separating a conducting medium from one of zero conductivity

Surface Characteristic Equation.

$$\frac{dV}{d\nu} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

or the component of flow at right angles to the surface is zero at every point on the surface.

If on the surface of separation between the media there be an electromotive force E acting from the medium of conductivity k_2 to that of conductivity k_1, we have besides (22) the equation

$$V_1 - V_2 - E = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where V_1, V_2 are the potentials at the point but on opposite sides of the surface of separation.

Electrostatic Analogy.

These differential equations are precisely similar to the equations obtained (pp. 102 *et seq.*) for electrostatic phenomena from the thermal analogy, and the solutions are, with the substitution simply of the analogues of certain quantities, at once interpretable for flow of electricity. These substitutions are specific conductivity for specific inductive capacity, flow of electricity per unit of area per unit of time for electrostatic induction, and line or tube of flow, for line or tube of force.

We shall consider in addition one or two simple and interesting cases.

Particular Cases of Flow.

1. An annular space contained within two cylindric surfaces is filled with a conducting liquid, and the inner and outer surfaces are maintained at given potentials: it is required to find the resistance of the liquid for conduction between these two surfaces.

Let the inner and outer radii of the space be denoted by r_1, r_2, and the distance of any point from the common axis by r. We may take the flow as everywhere radial between the two cylinders. Laplace's equation (21) becomes

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

By integration this gives

$$r\frac{dV}{dr} = A \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and

$$V = A \log r + B \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Hence if V_1, V_2 be the inner and outer potentials we have by (27)

$$V_1 = A \log r_1 + B,$$
$$V_2 = A \log r_2 + B;$$

Particular Cases of Non-Linear Flow.

and therefore

$$V_1 - V_2 = A \log \frac{r_1}{r_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

But if l be the length of the cylinder, V_1 the greater potential, and k the specific conductivity of the liquid, the total current is $-2\pi klr \,.\, dV/dr$ or $-2\pi klA$. Hence we have

$$\frac{V_1 - V_2}{-2\pi klA} = \frac{1}{2\pi kl} \log \frac{r_2}{r_1} \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The expression on the right is the total resistance of the liquid for conduction between the two cylinders, and depends only on the ratio of the two radii, and not on their absolute amounts. This is the case of the column of liquid between two coaxial cylindric plates in a voltaic cell. This result might have been obtained by interpretation from the equation for V, p. 56 above.

Small Spherical Electrodes in Infinite Medium.

2. Two small highly conducting spherical electrodes kept at different potentials are buried in an infinitely extending conductor of comparatively much lower specific conductivity k: it is required to find the resistance between the spheres.

The potential of each sphere must be nearly the same throughout its mass, and if the distance apart be great in comparison with the potential at any point in the neighbourhood of either, will be nearly in inverse proportion to the distance of the point from the centre of the sphere. Thus if V_1, V_2 be the potentials of the spheres in order of magnitude, and r_1, r_2 the radii, the potential at such a point will be V_1/r in one case and V_2/r in the other if r be the distance of the point from the sphere in question; and the corresponding outward gradients of potential dV/dr, dV/dr will be $-V_1/r^2, -V_2/r^2$. This gives at the surfaces of the electrodes the values $-V_1/r_1{}^2$ and $-V_2/r_2{}^2$. The outward flow from the sphere of higher potential is therefore $4\pi kV_1$, and the inward flow over the other $-4\pi kV_2$. Hence if γ be the total current, we have

$$\gamma = 2\pi k(V_1 - V_2).$$

For the total resistance R to conduction from one sphere to the other we get

$$R = \frac{V_1 - V_2}{\gamma} = \frac{1}{2\pi k} \quad . \quad . \quad . \quad . \quad . \quad (30)$$

a result independent of the radii of the spheres and of the distance between them. The result is of interest in connection

with the "earthing" of telegraph-wires and other conductors, for we infer that the resistance between two electrodes buried in the earth is practically independent of their distance apart.

If the conductor were separated into two parts by a plane passing through the centres of the spheres, the resistance between the hemispherical electrodes in each part would be double that given by (30), or $1/\pi k$.

Small Disc-Electrodes in Infinite Medium.

3. The same case as in 2, except that the electrodes are circular discs. Supposing, as before, the electrodes to be at a distance great in comparison with either disc, the distribution of potential in the medium surrounding either is the same approximately as it would be if the electrode were alone and charged in an infinite medium. Let r_1, r_2 be the radii of the discs, V_1, V_2 their potentials in order of magnitude. If σ be the electric surface density at any point on the surface of the disc at potential V_1, then the outward normal component of electric force $-dV/d\nu = 4\pi\sigma$. Hence integrating over the whole disc (both faces), and putting Q for the whole charge, $-\int ds.\, dV/d\nu = 4\pi Q$. But the total outward flow is $\gamma = -k\int ds.\, dV/d\nu = 4\pi k Q = 8kV_1r_1$ [by (55) p. 53 above]. In the same way from the other disc $\gamma = 4\pi kQ_2 = -8kV_2r_2$. Hence

$$\gamma = 4k(V_1r_1 - V_2r_2) = (V_1 - V_2)/R.$$

Since $V_1r_1 = -V_2r_2$ this gives

$$R = \frac{r_1 + r_2}{8kr_1r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

We infer that the parts of R due to the respective discs are $1/8kr_1$ and $1/8kr_2$.

If the discs lie in the bounding surface conduction takes place from only one face of each, and the value of R is twice that just obtained.

Correction for Massive Electrode joined to a Wire.

This result gives an inferior limit to the correction to be made on the resistance of a cylindrical wire which is joined to a large mass of metal.* Let the junction be made by a thin disc of perfectly conducting matter. The end of the wire will be brought to one potential, and therefore its conducting power up to the disc fully made use of. Hence an inferior limit to the correction is an addition of $1/4kr_1$ to the resistance, or if k' be the conductivity of the wire, of $\pi k'r/4k$ to the length. Lord Rayleigh has given $\cdot 8242\, k'r/k$ as a superior limit to the addition to the length.

* See Maxwell, *El. and Mag.* vol. i. pp. 396, 397 (sec. ed.).

SECTION II.

VARIABLE FLOW.

Variable Flow of Electricity

HITHERTO we have been dealing with cases in which the time-rate of variation of the electric flow is zero, and have seen that the theory of such cases is analogous to that of the steady flow of heat. We shall now consider cases of variable flow under the conditions stated above that no effect of electromagnetic or peristaltic induction is taken into account. The theory of such cases is also analogous to that of the flow of heat, in fact we have only to modify Fourier's solutions for the variable flow of heat to suit the particular electrical problems which it is most important to solve. The justification of this process is of course, as in other cases, to be found in the agreement of the results with those of experiment.

Problem of Single-wire subterrene or submarine Cable.

We consider first the following problem, which is that of a single-wire telegraph cable :—A homogeneous wire of uniform cross-section, covered uniformly with a coating of material of comparatively small conductivity the external surface of which is kept at zero potential, has one end brought to a potential fulfilling some specified law of variation and existing for a stated interval of time, while the other end is maintained at zero potential ; it is required to find the potential and current at any point in the wire at any specified instant.*

Linear Flow.

The equipotential surfaces in the wire, it is evident, will not differ sensibly from planes at right angles to the axis, and we may therefore take the potential as having the same value at every point of any such cross-section. By Ohm's Law the flux of electricity across a cross-section at which the potential is V is (p. 162 above) $-\ k\,dV/dx$. Let A, B, C be three cross-sections in the wire in the order from left to right, and let the distance of B from A, and of C from B, be very small and equal to $\frac{1}{2}\delta x$, so that δx is the distance of C from A. If dV/dx be the gradient of potential at B the gradient at A or at C is given by

$$\frac{dV}{dx} \mp \frac{1}{2}\frac{d^2V}{dx^2}\,\delta x,$$

* The electrical constants of the sending and receiving apparatus are here neglected. In practice, except in the case of a long cable, these constants must be taken into account. Some further treatment of this subject will be given in a later chapter.

according as the upper or lower sign is taken. Taking now for convenience k as the conductivity of the conductor per unit of length, we have by Ohm's law for the flow towards the right across A and C respectively the expressions

$$-k\left(\frac{dV}{dx}-\tfrac{1}{2}\frac{d^2V}{dx^2}\delta x\right),$$

$$-k\left(\frac{dV}{dx}+\tfrac{1}{2}\frac{d^2V}{dx^2}\delta x\right).$$

Leakage across Lateral Surface.

The flow across the outer surface is proportional to the difference of potentials between the wire and the external surface of the coating, that is to V. If we denote by h the conductivity of unit length of the coating, the time-rate of loss of charge across the lateral surface of the portion between A and C is $hV\delta x$. The total rate of loss of charge from this portion of the wire is equal to the excess of the rate of loss across C and the lateral surface above the rate of gain across A, and is therefore

$$-k\frac{d^2V}{dx^2}\delta x + hV\delta x.$$

The effect of loss of charge must be to lower the potential of the element between A and C, and the rate of fall of potential must be equal to the last expression divided by the electrostatic capacity of the element. By (60), p. 56, the capacity of the wire per unit of length, if of circular section and covered with a coaxial insulating coating also of circular section, is $1/2\log(b/a)$, where a is the internal, b the external radius of the covering.

Differential Equation of Potential, combining Linear Flow and Leakage.

Denoting this by c, we have for the capacity of the element $c\,.\,\delta x$. Dividing the rate of loss of charge by this number, and equating the result to $-dV/dt$, the time-rate of fall of potential, we get the differential equation

$$\frac{dV}{dt}=\frac{k}{c}\frac{d^2V}{dx^2}-\frac{h}{c}V. \quad \ldots\ldots \quad (32)$$

This is precisely the same as the equation of the linear motion of heat given by Fourier,* and his solutions are immediately applicable. It is of course for $h=0$ a particular case of (97), p. 105 above.

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* *Théorie Analytique de la Chaleur*, Chap. II. Art. 105.

This equation may be simplified by writing $V = \epsilon^{-ht/c}v$, when it becomes

$$\frac{dv}{dt} = \frac{k}{c}\frac{d^2v}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (32bis)$$

which is the differential equation for $h = 0$ or the case of zero leakage. A solution for the latter case can be converted into one for any value of h by simply multiplying the potential found by $\epsilon^{-ht/c}$.

Sir William Thomson has given the name *diffusivity* to the quantity k/c, and denotes it by κ. The quantity h corresponds to what for thermal radiation he has called *emissivity*.*

Cable of Finite Length. Integral for one end at constant potential, and the other at zero potential.

To integrate (32) for the case proposed above, let first the end, $x = 0$, of the cable, to which the battery is applied, be brought suddenly at time $t = 0$ to potential V_0 and kept at that potential ever after. Let x be measured from that end, and let l be the length of the cable. The potential for $x = l$ is likewise always zero; and there is a continual approach with lapse of time to a state of uniform variation of potential with resistance from one end to the other. These conditions are fulfilled and the differential equation satisfied by the solution

$$V = V_0 \frac{\epsilon^{(l-x)\sqrt{h/\kappa c}} - \epsilon^{-(l-x)\sqrt{h/\kappa c}}}{\epsilon^{l\sqrt{h/\kappa c}} - \epsilon^{-l\sqrt{h/\kappa c}}}$$

$$+ V_0\epsilon^{-ht/c}\sum_{i=1}^{i=\infty} A_i\epsilon^{-i^2\pi^2\kappa t/l^2}\sin\frac{i\pi x}{l}, \quad . \quad . \quad . \quad (33)$$

where i is any integer, and κ is written for k/c.

It only remains to determine the constant A_i, so that when t is only infinitesimally greater than 0, $V = V_0$ for $x = 0$, and $= 0$ for every other value of x. Putting $t = 0$ in (33) we get the equation

$$\sum_{i=1}^{i=\infty} A_i \sin\frac{i\pi x}{l} = -\frac{\epsilon^{(l-x)\sqrt{h/\kappa c}} - \epsilon^{-(l-x)\sqrt{h/\kappa c}}}{\epsilon^{l\sqrt{h/\kappa c}} - \epsilon^{-l\sqrt{h/\kappa c}}},$$

which must hold for every value of x greater than zero. Multiplying both sides by $\sin(j\pi x/l)dx$, where j is any integer, and integrating from $x = 0$ to $x = l$, we get on the left (since

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* *Encycl. Brit.*, Art. "Heat", § 71.

every term vanishes except that for which $j = i$) $A_i \, . \, l/2$; and on the right $-\, i\pi\kappa lc/(i^2\pi^2\kappa c + hl^2)$. Hence (33) becomes

Potential at distance x from sending end.

$$V = V_0 \frac{\epsilon^{(l-x)\sqrt{h/\kappa c}} - \epsilon^{-(l-x)\sqrt{h/\kappa c}}}{\epsilon^{l\sqrt{h/\kappa c}} - \epsilon^{-l\sqrt{h/\kappa c}}}$$

$$- 2V_0 \epsilon^{-ht/c} \sum_{i=1}^{i=\infty} \frac{i\pi\kappa c}{i^2\pi^2\kappa c + hl^2} \epsilon^{-i^2\pi^2\kappa t/l^2} \sin \frac{i\pi x}{l} . \quad . \quad (34)$$

The series on the right is convergent, and admits of easy numerical evaluation.

If the leakage is inconsiderable, that is, if h may be taken as zero, the equation reduces to

$$V = V_0 \frac{l - x}{l} - 2V_0 \sum_{i=1}^{i=\infty} \frac{1}{i\pi} \epsilon^{-i^2\pi^2\kappa t/l^2} \sin \frac{i\pi x}{l} . \quad . \quad (35)$$

From this we obtain the current γ at distance x from the end $x = 0$ (the sending end) by finding $-\, kdV/dx$. We thus get

Current at distance x from sending end.

$$\gamma = kV_0 \frac{\sqrt{h}}{\sqrt{\kappa c}} \frac{\epsilon^{(l-x)\sqrt{h/\kappa c}} - \epsilon^{-(l-x)\sqrt{h/\kappa c}}}{\epsilon^{l\sqrt{h/\kappa c}} - \epsilon^{-l\sqrt{h/\kappa c}}}$$

$$+ \frac{2kV_0}{l} \epsilon^{-ht/c} \sum_{i=1}^{i=\infty} \frac{i^2\pi^2\kappa c}{i^2\pi^2\kappa c + h^2l^2} \epsilon^{-i^2\pi^2\kappa t/l^2} \cos \frac{i\pi x}{l} . \quad . \quad (36)$$

When $x = l$ and $h = 0$, this becomes

$$\gamma = \frac{kV_0}{l} \left\{ 1 + 2 \sum_{i=1}^{i=\infty} (-1)^i \epsilon^{-i^2\pi^2\kappa t/l^2} \right\}, \quad . \quad . \quad (37)$$

a rapidly convergent series which gives the current at the end $x = l$ (the receiving end) when the leakage is zero.

The ordinates of curve A, Fig. 35, calculated from (37), give the values of γ for different values of t as abscissæ. The final value of the current is taken as unity; and a, the unit of the scale of abscissæ, is for the reason stated below made to represent $t = 2l^2/\pi^2\kappa \, . \log \frac{4}{3}$. The sum of the series on the right approaches $\frac{1}{2}$ as t is made more and more nearly zero. Hence the current at the end of the cable is, as was to be expected, zero immediately after the first contact. It does not differ sensibly from zero

Retardation of Signals.

until the first term of the series is greater than $\frac{3}{4}$, that is until t is greater than $l^2/\pi^2\kappa \,.\, \log \frac{4}{3}$. This value of t has been called by Sir William Thomson the retardation of the cable. After the interval of retardation has elapsed the current increases, as is shown in the diagram, rapidly at first and more and more gradually afterwards towards the value kV_0/l, which it reaches when $t = \infty$. This agrees with what we ought to expect, as k/l is the resistance of the whole cable and V_0 the difference of potential between its extremities: the state of the cable approaches a uniform gradient of potential from end to end.

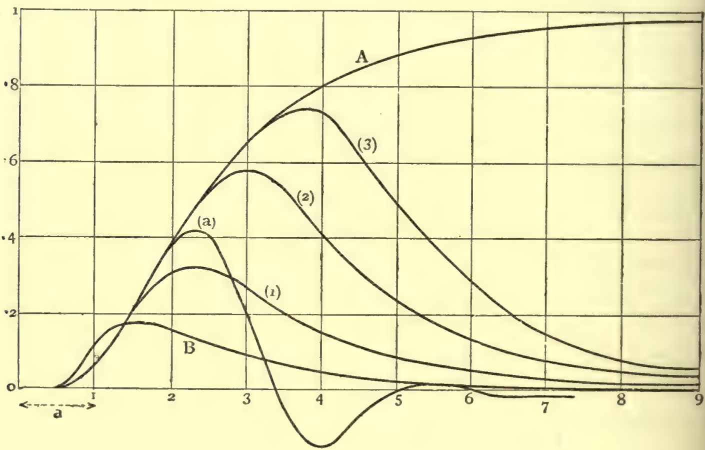

FIG. 35.

Graphic Solutions in different cases.

Curve (1) gives the current at the receiving end, for different values of t as abscissæ, in the case in which the sending end is brought suddenly to potential V_0, and maintained at that potential for an interval of time a, or twice the retardation, and then brought suddenly to zero potential and kept so ever after. It has been constructed by compounding with A the same curve as A drawn on the opposite side of the line of abscissæ and beginning at a distance a to the right of zero. Curves (2) and (3) give similarly the currents for the cases of contact at the same constant potential lasting intervals, respectively, four times and six times the retardation.

Curve (B) gives the current for the case of contact of duration τ infinitely short. The equation or a finite value of τ is evidently

Graphic Solutions in different cases.

$$u = \gamma_t - \gamma_{t-\tau}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

where γ_t denotes the current at time t due to $+ V_0$ established at the origin when $t = 0$, and $V_{t-\tau}$ the current due to $- V_0$ alone established at time $t = \tau$. But if τ is very small we may write

$$u = \tau \frac{d\gamma}{dt} = \frac{2kV_0\pi^2\kappa\tau}{l^3} \sum_{i=1}^{i=\infty} (-1)^{i+1}\, i^2\epsilon^{-i^2\pi^2\kappa t/l^2}. \quad . \quad (39)$$

This curve coincides with the other curves from $t = 0$ to $t = a/2$, and rises rapidly to a maximum which it reaches when $\pi^2\kappa t/l^2$ is $(\frac{3}{4})^3$ nearly.

Considering the distribution of potential in two cables of lengths l, l' with the same potential at the sending ends, we see that points x, x', fulfilling the condition $x/l = x'/l'$ will be at the same potential for the same value of t if $\kappa/l^2 = \kappa'/l'^2$, that is, the two cables will have the same retardation only if the diffusivities are as the squares of the lengths of the cables. But diffusivity $= k/c = l/rc$, where r is the resistance of unit length of the cable; hence in order that the retardation at distances x varying as the lengths of the different cables, or at the receiving ends, may remain constant rc must vary inversely as the square of the length of the cable. This means practically that the diameter of the conductor and the external and internal diameters of the covering must be doubled when the cable is doubled in length in order that the speed of signalling may remain unchanged.*

Conditions of similarity in two cables of different lengths.

* This conclusion was given by Sir William Thomson in the early days of Submarine Telegraphy, when the first Atlantic cable was being projected, and the recognition of its validity by those interested in such undertakings, led to the adoption of copper of the highest possible conductivity for the conductors of the cables, and of material combining low specific inductive capacity with high resistance for the covering. This has given rise to a new industry, the manufacture on a commercial scale of practically pure copper, now carried on to an enormous extent. The scientific experiments made in order to find suitable materials have added greatly to knowledge of electrical properties of bodies.

The same conclusion, it is to be remarked, may be derived from the differential equation directly. For the two equations

$$\frac{dV}{dt} = \frac{1}{rc}\frac{d^2V}{dx^2}, \quad \frac{dV'}{dt'} = \frac{1}{r'c'}\frac{d^2V'}{dx'^2}$$

are identical if $x^2/x'^2 = tr'c'/t'rc$; in other words two cables are at equal potentials at similarly situated points, (that is points for which $x/x' = l/l'$), at the same time, if $x^2/x'^2 = r'c'/rc$. This holds also in the case of leakage if the further relation $l^2/l'^2 = h/h'$ is fulfilled.

'Velocity of Electricity.'

It will be observed that there is properly speaking no velocity of propagation of electricity through a cable. In the examples given above (p. 170) as soon as contact is made at the sending end the potential at distance x begins to rise, infinitely slowly at first, but with gradually increasing rapidity until after a certain interval of time has elapsed the potential has risen to a specified fraction of its maximum amount. This interval of time depends as we have seen, p. 170, on the nature of the cable.*

In the case, however, of an impressed potential at the sending end, varying as a simple harmonic function of the time, there is a definite rate of propagation of the electric impulses through the cable. This case we shall consider later.

Infinitely Long Cable:

Equations (34) . . . (39) contain the complete solution of the problem proposed for the different circumstances considered; and from this solution we can easily obtain the solution of related problems of great interest. We shall consider first the case of an infinitely long cable, or which is the same, a cable whose length l is great in comparison with the distance x of any section considered; and we shall assume first that there is no leakage. From the solution on this supposition we can easily pass, as shown above, to that for an infinitely long cable with a covering of uniform conductivity h per unit of length.

Derivation of Solution from that for Finite Cable.

Assuming that l is very great in comparison with x we may write in (35) a for $i\pi x/l$, da for $\pi x/l$, and the sign of integration for that of summation. Thus we obtain

$$V = V_0 - \frac{2V_0}{\pi}\int_0^\infty \frac{1}{a}\,\epsilon^{-a^2\kappa t/x^2}\sin a\,da \quad . \quad . \quad . \quad (40)$$

This integral may be simplified as follows. It is proved in works on Integral Calculus that

$$\int_0^\infty \epsilon^{-a^2/4z^2}\cos na\,da = \sqrt{\pi}z\epsilon^{-n^2z^2}.$$

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* See 'Velocity of Electricity,' *Math. and Phys. Papers*, by Sir W. Thomson. Vol. ii. p. 131.

Multiplying both sides of this equation by dn and integrating with respect to n between the limits 0 and 1 (on the left within the sign of integration), we get

$$\int_0^1 \frac{1}{a} \epsilon^{-a^2/4z^2} \sin a\, da = \sqrt{\pi} \int_0^1 2\epsilon^{-n^2z^2}\, dn = \sqrt{\pi} \int_0^z \epsilon^{-y^2}\, dy.$$

Writing the last integral in the form $\int_0^z \epsilon^{-z^2}\, dz$, and putting $z^2 = x^2/4\kappa t$ at the superior limit we have

$$V = V_0 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^z \epsilon^{-z^2}\, dz\right) \quad . \quad . \quad . \quad . \quad (41)$$

Numerical Calculation of Potentials.

It has been shown by Sir William Thomson * that

$$\epsilon^{-z^2} = \frac{2\sqrt{\pi}}{a} \sum_{i=0}^{i=\infty} \epsilon^{-(2i+1)\pi^2/4a^2} \cos \frac{(2i+1)\pi z}{a} + R,$$

where i is an integer, and R a quantity which is insensible if a is so large that to the degree of accuracy to which the value of ϵ^{-z^2} is desired $\epsilon^{-a^2/4}$ may be neglected.

By multiplying both sides of this equation by $f(z)dz$ and integrating between 0 and z we can evaluate $\int_0^z \epsilon^{-z^2} f(z)dz$ in any case in which $\int_0^z f(z) \cos(i\pi z/a) \,.\, dz$ can be easily calculated. Putting $f(z) = 1$ we get

$$\int_0^z \epsilon^{-z^2}\, dz = \frac{2}{(2i+1)\sqrt{\pi}} \sum_{i=0}^{i=\infty} \epsilon^{-(2i+1)^2\pi^2/4a^2} \sin \frac{(2i+1)\pi x}{a} + S,$$

where S is insensible if a fulfils the condition already stated. The series is rapidly convergent, and a very few terms suffice to give the numerical value of the integral to a high degree of accuracy.

Calculating thus the values of V for different values of t and a single value of x, and plotting the results with values of t as abscissæ and the corresponding values of V as ordinates we get

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* *Math. and Phys. Papers*, vol. ii. p. 56, 'On the Calculation of Transcendents of the form $\int_0^x \epsilon^{-x^2} f(x)\, dx$.'

the curve A, Fig. 36, which therefore represents the rise of potential at the distance x from the origin after the establishment of potential V_0 at the origin.

Graphic Solutions for Short Contacts. If the potential at the origin be maintained at V_0 for an interval of time τ, and be then brought to zero and kept so ever after, the potential at any section at distance x may be supposed produced by supposing impressed at the origin a potential V_0 from $t = 0$ to $t = \tau$, and a potential $- V_0$ from $t = \tau$ to $t = \infty$. This may be supposed done by connecting for an interval τ, at the end, $x = 0$, of the cable one terminal of a suitable battery whose

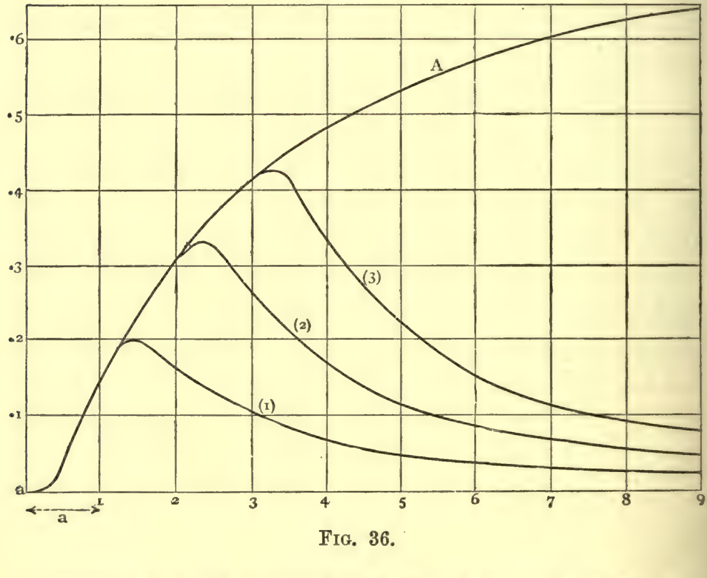

FIG. 36.

resistance is small compared with that of the cable, while the other terminal is connected with the earth, then disconnecting the battery and keeping both ends of the cable in contact with the earth. The curve of potential for the former of these is A, that for the latter is simply A drawn on the opposite side of the line of abscissæ and beginning at a distance τ to the right of zero. The resultant obtained by compounding these two curves shows for the case considered the variation of potential at distance

Graphic Solutions for Short Contacts.

x from the origin. Potential curves drawn in this manner are given in Fig. 36, (1) for $\tau = x^2/4\kappa$, (2) for $\tau = 2x^2/4\kappa$, (3) for $\tau = 3x^2/4\kappa$, that is for values of τ respectively a, $2a$, $3a$ on the scale of the diagram.

Curves can be drawn in a similar manner for other cases; for example the case of potential at the origin $+ V_0$ for an interval $3a$, then $- V_0$ for an equal interval, then $+ V_0$ for an interval a, and zero potential ever after. This curve would be drawn by compounding with the curve A, Fig. 36, three other curves, as follows,—a negative curve with ordinates double those of A for the same abscissæ, and starting at $t = 3a$, a positive curve precisely the same as the last in ordinates and abscissæ, and starting at $t = 6a$, and lastly, a negative curve precisely the same as A starting from $t = 7a$. Another important case is that in which the potential at the origin is V_0 for say $3a$, then $- V_0$ for $2a$ and zero ever after. The curve for this case would be drawn by omitting the last curve of the previous example, and making the positive curve start at $t = 5a$, instead of at $t = 6a$.

Curb-Signalling.

These examples are of interest as illustrating what is called "curb-signalling" through telegraph cables. When to produce a signal the battery is applied at the sending end of the cable for any interval, and that end then placed in contact with the earth as in ordinary uncurbed signalling, the potential at a distance x rapidly rises and then slowly falls. To more rapidly discharge the cable so as to bring the effect of one signal to zero before another is begun, and thus render the signals sharper and more distinct, the operator, instead of putting the cable to earth after the first application of the battery, reverses the battery on the cable, generally for a shorter interval, as in the second case just described, and then connects to the earth before beginning the next signal. This has been called signalling with *single-curb.* Instead however of thus dividing the signal into two parts, a positive and a negative, the operator may arrange to divide it into three parts, a positive, a negative and a positive, of suitable durations, say $2a$, $2a$, a, or $3a$, $2a$, a, and then connects to earth. The effect of the positive third part is to render the potential of the cable more nearly zero throughout at the end of the signal. This has been called signalling with *double-curb.** Curve (a) Fig. 35 shows the current in the cable for the case of curb-

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* An instrument by which the signals are made and the curb, either single or double, in any required proportions, is applied automatically, has been invented by Sir William Thomson. For description see *Journal of the Society of Telegraph Engineers* for 1876.

signalling in which the positive pole of the battery is applied for an interval $3a/2$, the negative for an equal interval, and lastly the positive for an interval $a/2$.

Analytical Solution for Infinitely Short Contact.

The equations of the curves (1), (2), (3) of Fig. 36 are easily obtained from (41) above. For let $V_{t-\tau}$ denote the potential at distance x from the origin due to potential V_0 established at the origin when $t = \tau$, and U the potential at x after time t, due to $+ V_0$ established at the origin when $t = 0$ with $- V_0$ superimposed when $t = \tau$. We have

$$U = V_t - V_{t-\tau} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42)$$

If τ be small we may write this equation

$$U = \tau \frac{dV}{at} = \tau \frac{dV}{dz} \frac{dz}{dt}.$$

Using this in (41) we get

$$U = \frac{V_0 x \tau}{2\pi^{\frac{1}{2}}\kappa^{\frac{1}{2}}t^{\frac{3}{2}}} \epsilon^{-x^2/4\kappa t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

If the interval τ be finite then plainly

$$U = \frac{V_0 x}{2\pi^{\frac{1}{2}}\kappa^{\frac{1}{2}}} \int_0^\tau \frac{1}{(t - \theta)^3} \epsilon^{-x^2/4\kappa(t-\theta)}\, d\theta \quad . \quad . \quad . \quad (44)$$

where θ is any value of t less than τ.

In actual practice the value of the potential at the origin is not constant throughout the whole interval during which the battery is applied but varies with the time. Hence the potential at the origin after the interval has elapsed from the instant at which the battery is applied may be denoted by $F(\theta)$. This substituted in (43) for V_0 gives

$$U = \frac{xF(\theta)\tau}{2\pi^{\frac{1}{2}}\kappa^{\frac{1}{2}}(t - \theta)^{\frac{3}{2}}} \epsilon^{-x^2/4\kappa t}, \quad . \quad . \quad . \quad . \quad . \quad (45)$$

and (44) becomes

$$U = \frac{x}{2\pi^{\frac{1}{2}}\kappa^{\frac{1}{2}}} \int_0^\tau \frac{F(\theta)}{(t - \theta)^{\frac{3}{2}}} \epsilon^{-x^2/4\kappa(t-\theta)}\, d\theta \quad . \quad . \quad . \quad (46)$$

It is to be noted that only values of t which are greater than τ can be used in the evaluated integrals of (44) and (46).

The current γ at time t and distance x from the origin can be

found in every case by calculating $-k.dV/dx$. For example from (41) we get

$$\gamma = -k\frac{dV}{dz}\cdot\frac{dz}{dx} = \frac{kV_0}{\pi^{\frac{1}{2}}(\kappa t)^{\frac{1}{2}}}\,\epsilon^{-x^2/4\kappa t}, \quad . \quad . \quad . \quad (47)$$

Calculation of Current for Long Contact.

which gives the current at distance x from the sending end in a cable which is so long that kV_0/l is negligible. The value of γ in this case is a maximum for $t = x^2/2\kappa$, and gradually falls to zero for $t = \infty$.

If l be not so great that kV_0/l can be neglected, the term $-V_0x/l$ must be restored to equation (41) before differentiation. The current in this case, at distance x small in comparison with l, is

$$\gamma = kV_0\left(\frac{1}{l} + \frac{\epsilon^{-x^2/4\kappa t}}{\pi^{\frac{1}{2}}(\kappa t)^{\frac{1}{2}}}\right), \quad . \quad . \quad . \quad . \quad (48)$$

and continually approaches the value kV_0/l. The character of the curve is the same as that of A, Fig. 35, above.

Current in case of Infinitely Short Contact.

The equation for γ in the case of an infinitely short contact is obtained from (43) as before by calculating $-k.dU/dx$. We get

$$\gamma = \frac{kV_0\tau}{2\pi^{\frac{1}{2}}\kappa^{\frac{1}{2}}t^{\frac{3}{2}}}\left(\frac{x^2}{2\kappa t} - 1\right)\epsilon^{-x^2/4\kappa t}. \quad . \quad . \quad . \quad . \quad (49)$$

This has a maximum for $t = x^2/12\kappa t$, is zero for $t = x^2/2\kappa$ and negative for greater values of t, and gradually approaches zero as t increases to ∞.

Signalling by Condensers.

In practice the speed of signalling is increased by the use of condensers, so that the conductor of the cable is kept insulated. One terminal of the battery is connected to earth, the other to one surface A of a large condenser, the other surface of which B is joined with the near end of the cable. The farther end of the cable is in contact with one surface B' of another large condenser of which the other surface A' is connected to earth. The surface A comes rapidly after contact to the full potential which can be produced by the battery, and the surface B becomes oppositely charged while the surfaces B' and A' at the farther end become electrified in the manner of A and B respectively. There is thus a (positive or negative) flow of electricity from the battery to A, from B to B' in the cable and from A' to earth, and this instead of continuing and approaching a steady state, as in the cases already considered, while the battery contact is maintained, falls off towards zero as the cable approaches a uniform potential throughout. The signals are thereby, when the operations

described above are performed at A, which is now the sending end, rendered sharper and a higher speed of signalling is obtainable.

Transmission of Electric Wave along a Submarine Cable.

We shall conclude this part of the subject with the solution for an infinitely long, well insulated submarine cable with a potential at the origin varying according to a simple harmonic function of the time. Let the potential be $V_0 \sin 2nt$, then the potential at distance x from the origin and time t measured from any instant at which the potential at the origin was zero will be given by the equation

$$V = V_0 \epsilon^{-x\sqrt{n/\kappa}} \sin (2nt - x\sqrt{n/\kappa}),$$

for this value of V satisfies the differential equation and all the other required conditions.

The interval between the time of any particular phase at the origin and that of the corresponding phases at distance x is $x/2\sqrt{n\kappa}$ and the corresponding value of V is diminished in the ratio of 1 to $\epsilon^{-x\sqrt{n/\kappa}}$. An electric pulse or wave of potential, of amplitude diminishing per unit of distance travelled in the geometrical ratio $\epsilon^{-\sqrt{n/\kappa}}$, is thus propagated along the cable with velocity $2\sqrt{n\kappa}$, that is the velocity of propagation is directly as the square root of the product of the frequency (n/π) of the oscillation and the conductivity of the cable, and inversely as the electrostatic capacity per unit of length.

Use of Telephone on Cable.

Such a harmonic variation of potential could be produced at the sending end by a telephone responding to a musical note of definite pitch. If the note have a frequency of 100, that is to say a period of 1/100 of a second, and the cable have a copper conductor of resistance of 5 ohms (5×10^{-11} C.G.S. electrostatic units) per knot, and an electrostatic capacity of ·3 microfarad per knot (3×10^4 C.G.S. electrostatic units) the velocity of propagation of the note will be about 930 knots per second; and the amplitude will be diminished to $\frac{1}{2}$ its initial value in traversing about 32 knots and to $\frac{1}{10}$ in traversing about 96 knots. In the case of a telephone responding to notes of different pitches produced simultaneously, the pulses corresponding to the higher notes would be transmitted with the greater velocities, and would be received in order of pitch beginning with the highest.*

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* The subject of the Transmission of Electrical Waves along a Wire, here only touched upon under limitations, has been more fully discussed by Professor J. J. Thomson, *Proc. London Math. Soc.* vol. xvii. Nos. 272, 273; and by Mr. Oliver Heaviside, *Phil. Mag.* Feb. 1888. See also Sir William Thomson's *Math. and Phys. Papers*, vol. ii.

CHAPTER III.

UNITS AND DIMENSIONS.

A PHYSICAL quantity is expressed numerically in terms of some convenient magnitude of the same kind taken as unit and compared with it. The expression of the quantity consists essentially of two factors, a *numeric*,* and the *unit* with which the quantity measured is compared; and the numeric is the ratio of the quantity measured to the quantity chosen as unit. Thus when a certain distance is said to be 25 yards, what is meant is that the distance has by some process

Two Factors of expression of a Physical Quantity.

Numeric and Unit.

* The term *numeric* has been introduced by Prof. James Thomson (Thomson's "Arithmetic," Ed. LXXII., p. 4) as an abbreviation of "numerical expression." It denotes a number, or a proper fraction, or an improper fraction, or an incommensurable ratio. We shall find it convenient to employ it here where we wish to lay stress on the fact that we are dealing with what are essentially numerical expressions. Of course what is actually meant by the conveniently brief expressions "a length, L," "a mass, M," "a force, F," and the like, is simply that L, M, F, &c., denote the numerics which express the respective quantities in terms of the units chosen, that is, are, as we shall say below, the *numerics of the quantities* in terms of those units. Further in such phrases as "the product of mass and velocity," or "the product of charge and potential," and so on, the product (or whatever other function is specified) of the numerics is of course what is intended. If all such expressions were made verbally unexceptionable, the resulting prolixity would be intolerable.

Numeric and Unit.

been compared with the length, under specified conditions, of a certain standard rod (which length is defined as a yard) and the ratio of the former to the latter found to be 25.

The unit of measurement is of course itself capable of being expressed numerically in terms of any unit of the same kind, and in the same way therefore its full expression consists of a numeric and the new unit. Hence if N be the numeric of any physical quantity in terms of the unit, N' in terms of another unit, and n the numeric of the first unit in terms of the second, we have

$$N' = n . N \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Change-Ratios.

In order therefore to find the expression N' of the quantity in terms of the second unit from its expression N in terms of the first we have to multiply by n, the ratio of the first unit to the second. This numeric has been appropriately called the *change-ratio* for the change from the first unit to the second.

Arbitrary Units.

The change from N to N' cannot be made unless the change-ratio n, is known. Each unit may have been arbitrarily chosen without reference to any other unit, and n determined by some process of measurement; or the units may have been derived from certain chosen fundamental units, and the ratio deduced from the relation of one system of fundamental units to the other. In the measurements described in this work the units employed are entirely of the second kind here referred to.

In the early days of electrical measurements the units

in use were most of them thus arbitrarily chosen, each without reference to other physical quantities; and further each investigator had his own standards, adopted to suit his own convenience, by which he tested the electrical qualities of the substances he employed. The great inconvenience, loss, and uncertainty caused by this want of a common system of measurement became intolerable when practical applications of electricity like those of submarine telegraphy began to be proposed and undertaken, and led to the adoption of so-called *absolute* units, that is, derived units depending on a system of fundamental units in no way affected by locality or other conditions of experimenting, or related to the instruments and materials of any investigator.

Arbitrary Units.

"Absolute" Units.

The system chosen is one which was suggested first by Gauss, and carried out by him to some extent for dynamical and for electric and magnetic quantities, extended and developed first by Wilhelm Weber, and then by Thomson, Maxwell, and others who formed the B. A. Committee on Electrical Standards, and gradually adopted, until finally, at the Congresses of Electricians held at Paris in 1881, 1882, and 1884, electric units founded on it were chosen for use by the whole civilised world. In this system the units of length, mass, and time are defined and taken as fundamental units, and from these, units for the measurement of all physical quantities are derived in the manner explained below.

Gauss's suggested System

carried out by B.A. Committee.

It is to be noticed that although this system is an "absolute" system in the sense just stated, it is only one of several systems absolute in the same sense, which

might be constructed. It has, however, the great advantage that the fundamental units adopted are units of those physical quantities which are measured continually in the ordinary business of life, as thus all the derived physical units are brought into direct comparison with the standards of length and mass in the comparison and reproduction of which so much scientific labour has been expended, and with the elaborately-accurate measurements of time furnished by the astronomical observatories of the world.

"Dimensions" of Physical Quantities.

The task before us is to determine the manner in which the various derived units involve the fundamental units, that is, we have to determine for each quantity (p. 180 above) the change-ratio n in terms of the fundamental units. The formula which expresses n for a unit of measurement of any quantity we shall call the *Formula of Dimensions* or the *Dimensional Formula* of the quantity. To prevent the necessity for the constant repetition of these terms we shall denote the dimensional formula of any quantity, of which the numeric is denoted by any particular symbol, by the same symbol inclosed in square brackets. Thus we denote the dimensional formula of the quantity Q by the symbol $[Q]$.

Examples of the values of $[Q]$ will be found in dealing with the various units to which we now proceed. We shall first consider the definitions and relations of the fundamental units in common use and the derivation from them of the units of other physical quantities. In doing so we shall find the dimensional formula in each case and its numerical values for certain changes of units.

Fundamental Units.

(1) *Length.* The standard unit of length in Great Britain is defined by Act of Parliament in the following terms:* "The straight line or distance between the centres of the transverse lines in the two gold plugs in the bronze bar deposited in the Office of the Exchequer shall be the genuine standard of length at 62° F., and if lost it shall be replaced by means of its copies."

Fundamental Units adopted. Standards of Length.

Authorised copies are preserved at the Royal Mint, the Royal Society of London, the Royal Observatory at Greenwich, and the New Palace of Westminster. The comparison of the length of the standard with the lengths of its copies has been effected with the utmost scientific accuracy, and formed a most elaborate and important scientific investigation.

Imperial Yard defined.

The length of a simple pendulum which beats seconds has been determined for several places by means of very careful observations, and repeated pendulum experiments at these places would in the event of the destruction of the standard and all its copies give a means of accurately renewing them.

In France and in most Continental countries the standard of length is the *Metre*. This is defined as the distance between the extremities of a certain platinum bar when the whole is at the temperature 0° of the Centigrade scale. This rod was made of platinum by Borda, and is preserved in the national archives of France. As in the case of the yard, authorised copies

Metre defined.

* 18 and 19 Vict. c. 72, July 30, 1855.

whose lengths have been carefully compared with the standard are preserved in various places.

Relation of Metre to Dimensions of the Earth.

The metre was constructed in accordance with a decree of the French Republic passed in 1795,* which enacted, on the recommendation of a Committee of the French Academy of Sciences, consisting of Laplace, Delambre, Borda, and others, that the unit of length should be one ten-millionth part of the distance, measured along the meridian passing through Paris, from the Equator to the North Pole. The arc of that meridian extending between Dunkirk and Barcelona was measured by Delambre and Méchain, and from their results the standard metre was realised in platinum by Borda. The metre, it is to be observed, is not now defined in relation to the earth's dimensions, and later and more accurate results of geodesy have therefore not affected the length of the metre, but are themselves expressed in terms of the length which Borda's rod has at 0° C.

Decimal Multiples and Sub-Multiples of Metre.

In the French system the decimal mode of reckoning has been adopted for multiples and sub-multiples of all the units. Thus the metre is divided into ten equal parts each called a decimetre, the decimetre into ten parts each called a centimetre, and the centimetre into ten parts each called a millimetre. Again, a length of ten metres is called a decametre, of one hundred metres a hectometre, and one thousand metres a kilometre.† Of these, in accordance with the prevailing practice of scientific experimenters who adopted the suggestions of

* Loi du 18 germinal, an iii.

† See Table at end of this volume.

the B. A. Committee, the centimetre has been very generally chosen as the unit of length for the expression of scientific results, and on it as unit of length the electric and magnetic units approved by the International Congress of Electricians held at Paris in 1882 have been founded. The reason for this choice will appear when we consider the unit of mass.

Dimensional Formula of Length.

We shall denote the numeric of a length by L. The dimensional formula is therefore $[L]$.

For example, if we wish to find from the numeric of a length in terms of the yard as unit the numeric of the same length in terms of the metre as unit, we have $[L] = \cdot 91439$, the ratio of one yard to one metre; and this of course is equal to 36/39·3704, or 91·439/100, &c., the ratios of the numerics of the two units directly obtained according as the inch, or the centimetre, &c., is taken as unit of comparison. Similarly the value of $[L]$ for a change from the foot as unit to the centimetre as unit is 30·47945.

Standard of Mass.

(2) *Mass.* The legal standard of mass in Great Britain is the Imperial standard pound avoirdupois, a piece of platinum marked "P. S. 1844, 1 lb.," preserved in the Exchequer Office. In the Act of Parliament (the Act already referred to) which gives authority to the standard, it is called the "legal and genuine standard of weight;" and the Act provides that if the standard is lost or destroyed it may be replaced by means of authorised copies, which are kept in the same national repositories as the copies of the standard of length.

Imperial Pound defined.

It is to be noted that the word "weight" used in the

Ambiguity of word "Weight."

Act, is one which is constantly used in two distinct senses: (1) as here, to signify the quantity of matter in a body; (2) in its proper sense, to signify the downward force of gravity on the body. It is evident that these two senses are distinct. The quantity of matter in a body is invariable; the force of gravity upon the body depends on the situation of the body, and may even be zero. At a given place the forces of gravity on different bodies are, as was proved by Newton by pendulum experiments, proportional to their masses, and thus a comparison of the weights of different bodies gives a direct comparison of their masses.

The pound has been generally used in Great Britain as the unit of mass for the expression of dynamical results, but in engineering and the arts, larger units, for example, the ton, or mass of 2240 lbs., and the hundred-weight, or mass of 112 lbs., are frequently employed.

Kilogramme defined.

The French standard of mass is a piece of platinum called the *Kilogramme des Archives,* made also by Borda in accordance with the decree of the Republic mentioned above. It was connected with the standard of length by being made a mass as nearly as possible equal to that contained in a cubic decimetre of distilled water at the temperature of maximum density, 4° C. The comparison was of course made by weighing, and so far as this process was concerned it was possible to obtain great accuracy, but the density of water is somewhat difficult to determine with exactness, and is still in a small degree uncertain. The relation between the standards is, however, so nearly that stated above that,

for practical purposes, the error may be neglected. But on account of this uncertainty it is important to remember that the standard is *defined* as the kilogramme made by Borda, and not as the mass of a cubic decimetre of distilled water at 4° C., which it approximately equals.

A comparison between the French and British standards of mass made by Professor W. H. Miller gave the mass of the Kilogramme des Archives as 15432·34874 grains.

Relation of French Standard of Mass to Standard of Length.

The gramme, defined as 1/1000 of the Kilogramme des Archives, and approximately equal to the mass of one cubic centimetre of water at 4° C., was recommended by the B. A. Committee as the unit of mass for the expression of experimental results generally, and this choice has now been ratified by the general practice of scientific men. The convenience of the adoption of this unit of mass lies in the fact that it is approximately the mass of unit volume of the substance, (water at its temperature of maximum density), usually taken as standard of comparison in the estimation of specific gravities of bodies, which therefore become in this case the same numbers as the densities of the bodies.

The multiples and sub-multiples of the gramme proceed decimally, and are distinguished by the same prefixes as those of the metre.*

Dimensional Formula of Mass.

We shall denote the numeric of a mass by M, and hence its dimensional formula by $[M]$.

The value of $[M]$ for a reduction from the pound as unit to the gramme as unit is 453·593, for a reduction from the grain as unit to the gramme as unit 15·432.

* See Table at end of this volume.

Time. (3) *Time.* The definition of equal intervals of time belongs to dynamics and cannot here be entered on, but according to it the times in which the earth turns through equal angles about its axis are to a very high degree of approximation equal. These intervals correspond to equal intervals of time shown by a correct clock, and if the clock just goes twenty-four hours in the time of an exact revolution of the earth about its axis (or, which is the same, the interval between two successive passages of a fixed star in the same direction across the meridian of any place), showing 0h. 0m. 0s. each time a certain point of the heavens called the First Point of Aries crosses the meridian in the same direction, it is said to show sidereal time.

Though sidereal time is used in astronomical observatories, it is more convenient in ordinary civil affairs to use solar time; but as the actual solar day, the interval between two successive transits of the sun across the meridian of any place, varies in length during the year, the standard interval is the mean of such intervals, and is called a *mean solar day.* On account of the orbital motion of the earth the mean solar day is about 3m. 55·9s. longer than the sidereal day.

Unit of Time defined. The mean solar second, defined as 1/86400 part of the mean solar day, is taken as the unit of time for the expression of all scientific results.

Other suggested Standards of Length, Mass, and Time. We have seen that the choice of the fundamental units is entirely arbitrary, and there is nothing in their nature which entitles them in any just sense to the term "absolute." The unit of time, which is based on

Other suggested Standards of Length, Mass, and Time.

the period of rotation of the earth, is, we have reason to believe, subject to a slow progressive lengthening, due to tidal retardation of the earth's rotation, and possibly also to frictional resistance of the surrounding medium; and as a matter of definition, without reference in all cases to realisation, it would be easy to find many much more satisfactory standards. Thus it has been suggested by Sir William Thomson * that the period of vibration of a metallic spring, and kept in a hermetically sealed exhausted chamber at a constant temperature, or the period of a particular mode of vibration of a quartz crystal (or other crystal of definite composition) of a specified size and shape and at a given temperature, would be theoretically preferable to the mean solar second, as fulfilling with a much nearer approach to perfection the condition of constancy. Clerk Maxwell † has also suggested as units of time the period of vibration of a gaseous atom of a widely diffused substance easily procurable in a pure form; or the period of revolution of an infinitesimal satellite close to the surface of a globe of matter at the standard density, which may be any density determined by a definite physical condition of any substance, for example the maximum density of water. This period is independent of the size of the globe ‡ and it has been pointed out that this advantage would also be obtained by founding

* *Electricity and Magnetism*, vol. i. p. 3; Thomson and Tait, *Nat. Phil.* vol. i. part i. p. 227 (second edition).

† *Ibid.*

‡ For water at the temperature of maximum density, it is approximately 10h. 3m.

Other suggested Standards of Length, Mass, and Time.

the unit of time on the period of a small simple harmonic vibration of a globe of standard density.

The mass of a gaseous atom, for example that of a sodium or hydrogen atom, and the wave-length of a definite line in the spectrum of an easily obtainable substance, for example the D lines in the spectrum of sodium, might be chosen as the units of mass and length. These units would be quite definite since, according to the kinetic theory of gases, the atoms of any one substance are undistinguishable from one another by any physical test.

Derived Units.

Dimensional Formula of Derived Units.

Let us suppose that the numeric N of a physical quantity is given by the equation,

$$N = C.L_1^{\,l} M_1^{\,m} T_1^{\,n} \,.\, L_2^{\,p} M_2^{\,q} T_2^{\,r} \,.\, \text{\&c.}, \quad . \quad . \quad (2)$$

where L_1, L_2, &c., M_1, M_2, &c., T_1, T_2, &c., are numerics of different lengths, masses, and times in terms of a certain chosen unit for each, and C is a numerical multiplier (generally equal to unity) which does not depend on the units adopted. Now let other units of length, mass, and time be chosen and let N' be the numeric of the same quantity in terms of these units, and L'_1, L'_2, &c., M'_1, M'_2, &c., T'_1, T'_2, &c., those of the lengths, masses, and times. Then we have

$$N' = C.L'^{\,l}_1 M'^{\,m}_1 T'^{\,n}_1 \,.\, L'^{\,p}_2 M'^{\,q}_2 T'^{\,r}_2 \,.\, \text{\&c.} \quad . \quad (3)$$

But by equation (1) $L'^l = L^l\,[L]^l, M'^m = M^m\,[M]^m$, and so on. Hence (3) becomes,

Dimensional Formula of Derived Units.

$$N' = C.L_1{}^l\,M_1{}^m\,T_1{}^n\,.\,L_2{}^p\,M_2{}^q\,T_2{}^r\,.\,\&c.\;[L]^{l+p+\&c.}$$
$$[M]^{m+q+\&c.}.\,[T]^{n+r+\&c.} \quad . \quad . \quad . \quad (4)$$

By equation (1) therefore the dimensional formula $[N]$ of the quantity is $[L]^{l+p+\&c.}.\ [M]^{m+q+\&c.}.\ [T]^{n+r+\&c.}$ In accordance with the notation $[N]$, we shall denote this in future by the more convenient expression $[L^{l+p+\&c.}.\ M^{m+q+\&c.}.\ T^{n+r+\&c.}]$.

Dimensions of Units.

The numerics $l+p+\&c.$, &c., correspond to what Fourier * called *les exposants des dimensions* of the quantities which entered into his analysis, and it is these numerics, not the dimensional formulas, which are properly the "dimensions" of the units. It was pointed out by Fourier that in equations involving the numerics of physical quantities every term must be of the same dimensions in each unit, otherwise some error must have been made in the analysis. This consideration affords in physical mathematics a valuable check on the accuracy of algebraic work.

It is obvious from equations (1) or (4) that the dimensional formula of the product of any number of numerics N_1, N_2 of different physical quantities is the product $[N_1.\ N_2.\ \&c.]$ of their dimensional formulas, and more generally that the dimensional formula of the product $[N_1{}^{\mu_1}.\ N_2{}^{\mu_2}.]$ &c., of any powers whatever of these expressions, is the product of the same powers of the corresponding dimensional formulas.

* *Théorie Analytique de la Chaleur*, Chap. II. Sect. IX.

Dimensions of Units.

We are now prepared to find the dimensional formulas of the various derived units. The process will consist in finding for each quantity the formula corresponding to the right-hand side of (2), and thence deriving according to (4) the proper formula of dimensions. We shall consider first the units of Area, Volume, and Density; then the various dynamical units which are involved in those of electrical and magnetic quantities.

Area.

Area. The general formula for the area of any surface can be put in the form CL^2, where L is a numeric expressing a length, and C is a numeric which does not change with the units. Hence by (4) the formula of dimensions for area is $[L^2]$.

Volume.

Volume. Similarly the formula for the numeric of a volume can be written CL^3, and the formula of dimensions is $[L^3]$.

Density.

Density. The *Density* of a body is expressed by the numeric of the mass per unit of volume. We shall denote it by the symbol D.

If the body be of uniform density, the numeric is obtained by finding the mass contained in any given volume of the body: the ratio of the numeric of the mass to the numeric of volume is the density.

If the body be of varying density, the density at any point is the limit towards which the ratio of the numeric of the mass contained in an element of volume including the point, to the numeric of the volume, approaches as the element is taken smaller and smaller. Thus if δV be an element of volume including a point at which the

density is D, and δM be the mass of the element, we have Density.

$$D = \frac{dM}{dV}.$$

In either case we have for the numeric of the volume taken CL^3, and for that of the mass contained in it M. Hence

$$[D] = ML^{-3}].$$

The *Specific Gravity* of a body is the ratio of the density of the body to the density of the standard substance, and is therefore a numerical ratio independent of the system of units adopted, that is, its dimensional formula is 1. If G denote the specific gravity of a body whose density is D, and D_s be the density of the standard substance, Specific Gravity.

$$D = G \,.\, D_s.$$

In the French system of units D_s is taken as unity and we have $D = G$. This is one great convenience of the French units of length and mass; but it is to be remembered that Density and Specific Gravity are essentially different ideas, and only coincide in numerical value when $D_s = 1$.

Dynamical Units.

Velocity. The velocity of a body is measured by the numeric of the length described per unit of time. Its specification involves direction as well as magnitude; but in dealing with the dimensions of velocity we are only concerned with the latter element. Velocity.

If the velocity is uniform, its numeric is the ratio of the numeric L of any distance traversed to the numeric T of the time in which it is described.

Velocity. If the velocity is variable, its numeric v at any instant is the value towards which the ratio of the numeric δL of the distance traversed in an interval of time including the instant, to the numeric δT of the interval, approaches as the interval is taken smaller and smaller. Hence

$$v = \frac{dL}{dT} = \dot{L},$$

where $\dot{L}$ denotes in Newton's fluxional notation the time-rate of variation of L. We shall use this symbol for velocity.

We see that the numeric of a velocity is the ratio of the numeric of a length to the numeric of a time-interval, and therefore we have

$$[\dot{L}] = [LT^{-1}].$$

As multiplier for a change from mile-minute units to centimetre-second units we have

$$n = 5280 \times 30{\cdot}4797/60 = 2682{\cdot}2136.$$

In statements of amounts of velocities there ought clearly to be a distinct reference to the unit of time: thus the expressions one mile per minute, 88 feet per second, 2682·2136 centimetres per second, are perfectly definite, and express the same velocity, while such a statement as a "velocity of 88 feet" is devoid of meaning.

Acceleration. The acceleration of a body is expressed by the numeric of the change of velocity per unit of time. Accelera-tion.

Like velocity, acceleration involves in its signification the idea of direction as well as of magnitude; and it is through a want of clear apprehension of this fact that difficulty is found by students in the theory of curvilinear motion.

Let $\delta\dot{L}$ be the velocity given in direction and magnitude which compounded with the velocity $\dot{L}$ which a particle possesses at the beginning of an interval of time δT would give the velocity in direction and magnitude at the end of that interval, then $\delta\dot{L}/\delta T$ is the *average* acceleration during that interval, and the limit towards which this ratio approaches as δT is made smaller and smaller is the true value of the acceleration at the beginning of the interval. That is, we have

$$\text{Acceleration} = \frac{d\dot{L}}{dT} = \ddot{L},$$

where $\ddot{L}$ denotes in the fluxional notation the time-rate of variation of $\dot{L}$, that is, of velocity.

We shall use this symbol for acceleration, and the two dots above the L will serve to recall the double reference to time which is plainly involved in the notion of acceleration. This should be clearly expressed in statements of amounts of acceleration. Thus such a statement as an acceleration of 981 centimetres per second per second, or 32 feet per second per second, is perfectly definite, while such phrases as an "acceleration of 981 centimetres" or "an acceleration of 32 feet per second," which are often used, are meaningless.

The dimensional formula is

$$[\ddot{L}] = [LT^{-2}].$$

For a change from mile-minute units to centimetre-second units,

$$n = 5280 \times 30{\cdot}4797/60^2 = 447{\cdot}0356.$$

Momentum.

Momentum. Taking for simplicity the case of a rigid body moving without rotation, that is, so that each particle of the body has the same velocity at the same instant, the momentum of the body is expressed as the product of the numerics of the mass of the body and its velocity. Hence it is expressed symbolically by $M\dot{L}$. The dimensional formula is therefore

$$[M\dot{L}] = [MLT^{-1}].$$

Rate of Change of Momentum.

Time-Rate of Change of Momentum. If the momentum of the body be not constant, then, since we suppose the mass constant, we must have for the time-rate of variation the expression $M\ddot{L}$, that is, the product of the numerics of the mass and the acceleration. The dimensional formula is therefore

$$[M\ddot{L}] = [MLT^{-2}].$$

Force.

Force (F). A force acting on a body is proportional to the time-rate of change of momentum. Hence the dimensional formula just found is that of force.

According to the system suggested by Gauss, a force is measured by the time-rate of change of momentum, that is, the constant, C, of equation (3) is in this case, as in the other cases we have considered, taken equal to unity. Unit force is therefore that force which acting

for unit of time on unit mass produces unit change of velocity, or simply that which produces unit acceleration in unit mass. Kinetic Unit of Force.

When the unit of mass is one pound, the unit of length one foot, and the unit of time one second, then unit force is that force which acting for one second on a pound of matter generates a velocity of one foot per second. This unit force has been called a *poundal.*

The unit force in the C.G.S. system, is that force which, acting for one second on one gramme of matter, generates a velocity of one centimetre per second. To this unit force the name *dyne* has been given. C.G.S. Unit of Force.

This method (sometimes called the *kinetic* method) of measuring forces has now superseded, for scientific purposes, the gravitation system formerly in use. In that system the unit of force is the force of gravity on the unit of mass, and has, therefore, different values at different places on the earth's surface, and at different vertical distances from the mean surface level. This substitution of an invariable unit of force, depending only on the standards adopted for length, mass, and time, instead of the former variable unit, is at the foundation of the system of units of measurement established by Gauss. It is to express this fact of invariability with locality and other circumstances that, as already explained, the term "absolute" is used for the unit of force and other derived units in this system.

Work (W). In dynamics *work* is said to be done *by* a force when the place of application of the force receives a component displacement *in the direction in* Work.

Work. *which the force acts,* and the work done by it is equal to the product of the force and the distance through which the place of application of the force has moved in that direction. The time-rate at which work is done by a force at any instant is therefore equal to the product of the force and the component of velocity in the direction of the force at that instant. The work done in overcoming a resistance through a certain distance is equal by this definition to the product of the resistance and the distance through which it is overcome. Among engineers in this country the unit of work generally used is *one foot-pound,* that is, an amount of work equal to that done in lifting a pound vertically against gravity through a distance of one foot. The weight of a pound of matter being generally different at different places, this unit of work is a variable one, and is not used in theoretical dynamics. In the absolute C.G.S. system of units, the unit of work is the work done in overcoming a force of one dyne through a distance of one centimetre, and is called one centimetre-dyne or one *erg.*

In practical electricity 10^7 *ergs* is frequently used as unit of work, and is called a *Joule.*

If F denote the numeric of a force and L the numeric of the space through which it has acted, the numeric of the work done is FL. Hence we have

$$[W] = [FL] = [ML^2T^{-2}].$$

Activity. *Activity* (A). The single word *Activity* has been used by Sir William Thomson as equivalent in meaning to "time-rate of doing work," or the rate per unit of time at which energy is given out by a working system; and

to avoid circumlocutions in what follows we shall frequently use the term in that sense. Among engineers in this country the unit rate of working is *one horse-power*, that is 33,000 foot-pounds per minute.

Unit Activity in the C.G.S. system is one *erg* per second. In practical electricity an activity of 10^7 *ergs* per second is frequently employed as unit. This unit has been called a *Watt*. Unit Activity.

Since Activity is measured by the numeric of the work done per unit of time, its dimensional formula is given by

$$[A] = [ML^2T^{-3}].$$

Energy (E). When a material system in virtue of stresses between its own parts and those of bodies external to it does work or has work done upon it, in passing from one state to another, it is said to give out or to gain energy. The energy given out or gained is measured by the work so done. Energy.

If the change be a change of motion, then, according as energy is given out or gained by the system, it is said to lose or gain *kinetic energy*. If the change be of any other kind, which can be classed under change of configuration, then, according as the system gives out or gains energy, it is in general said to lose or gain *potential energy*.

When we consider the work done by mutual forces between different parts of the same system, a loss of kinetic energy in the system is accompanied by an equal gain of potential energy, and *vice versâ*, so that the total energy of the system remains unchanged in amount. This is the principle called the *Conservation of Energy*.

Energy is measured by the same units as work, and its dimensional formula is the same as that of work, that is

$$[E] = [ML^2T^{-2}].$$

We shall here, for the sake of illustration, give three examples of the application of dimensional formulas to the solution of problems regarding units. The problems are taken from Professor Everett's *Units and Physical Constants.*

Examples of Problems in Units.

Ex. 1. If the unit of time be the second, the unit density 162 lbs. per cubic foot, and the unit of force the weight of an ounce at a place where the change of velocity g produced by gravity in one second is 32 feet per second, what is the unit of length?

Here the change-ratio by which we must multiply the numeric of the density of a body in the system of units proposed, to find its density in terms of the pound as unit of mass, and the foot as unit of length, is 162. We have therefore, omitting the brackets in the dimensional formulas,

$$ML^{-3} = 162,$$

where M is the number of pounds equivalent to the unit of mass, and L the number of feet equivalent to the unit of length. Also, it is plain that the unit of force in the proposed system is two foot-pound-second units. Hence we have also, since $T = 1$,

$$MLT^{-2} = ML = 2.$$

By division therefore we get $L^4 = 1/81$ or $L = 1/3$. The unit of length is therefore 4 inches.

Ex. 2. The number of seconds in the unit of time is equal to the number of feet in the unit of length, the unit of force is 750 lbs. weight (g being 32), and a cubic foot of the substance of unit density contains 13,500 ounces. Find the unit of time. Examples of Problems in Units.

Using M and L as in the last problem, and putting T for the number of seconds equivalent to the unit of time, we have plainly

$$ML^{-3} = \frac{13500}{16}$$

and

$$MLT^{-2} = 750 \times 32.$$

Therefore by dividing, and remembering that $L = T$, we get

$$T^2 = \frac{32 \times 750 \times 16}{13500} = \frac{16^2}{3^2}$$

That is the unit of time is $5\frac{1}{3}$ seconds.

Ex. 3. When an inch is the unit of length and T seconds the unit of time, the numeric of a certain acceleration is a; when 5 feet and 1 minute are the units of length and time respectively, the numeric of the same acceleration is $10a$. Find T.

The change-ratio or value of LT^{-2} for reduction to foot-second units is plainly in the first case $T^{-2}/112$, in the second $5/3600$. We get therefore

$$\frac{1}{12}T^{-2}a = \frac{5}{3600} \times 10a,$$

or

$$T = \sqrt{6}.$$

Derived Electrical Units. Electrostatic System.

Quantity of Electricity.

Quantity of Electricity [q]. In what is called the *electrostatic* system of units, which we here consider, and which is most convenient when electrostatic results independently of their bearing on electromagnetic phenomena only are required, the units of all the other quantities are founded on the definition of unit quantity of electricity given above (p. 3). This definition is, as we shall see, precisely similar to the definition of magnetic pole which forms the basis of another system of units called the *electromagnetic* system, of much wider and more important application than the electrostatic. Hence by Coulomb's law (p. 2) that electric attractions and repulsions are directly as the product of (the numerics of) the attracting or repelling quantities, and inversely as the second power of (the numeric of) the distance between them, if a quantity of positive electricity expressed by q be placed at a point distant L units from an equal quantity of positive electricity, the numeric F of the force between them is q^2/L^2. We have therefore the equation $q^2 = FL^2$, and therefore [q] is [$F^{\frac{1}{2}}L$] or [$M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-1}$].

Electric Surface Density.

Electric Surface Density [σ]. The density of an electric charge is (p. 7) measured by the quantity of electricity per unit of area. Therefore [σ] is [qL^{-2}] or [$M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}$].

Electric Force.

Electric Force and Intensity of Electric Field [i]. The electric force at any point in an electric field is (p. 6) the force with which a unit of positive electricity would

be acted on if placed at the point. Hence if the numeric of the quantity of electricity at a point P be q, and that of the electric force at that point be i, the numeric F of the force on the electricity is qi, and we have the equation $i = Fq^{-1}$. Therefore $[i]$ is $[Fq^{-1}]$ or $[M^{\frac{1}{2}}L^{-\frac{1}{2}}T^{-1}]$. Field Intensity.

The intensity of an electric field at any point is measured by the electric force at that point, and therefore has the same dimensional formula.

Electric Potential (V). The difference of potential between two points is (p. 7) measured by the work which would be done if a unit of positive electricity were placed at the point of higher potential and made to pass by electric forces to the point of lower potential. Hence in transferring q units of electricity through a difference of potentials expressed by V, an amount of work is done of which the numeric W is qV. We have therefore $V = Wq^{-1}$, and hence $[V]$ is $[Wq^{-1}]$ or $[M^{\frac{1}{2}}L^{\frac{1}{2}}T^{-1}]$. Electric Potential.

Capacity of a Conductor (K). The capacity of an insulated conductor is the quantity of electricity required to charge the conductor to unit potential, all other conductors in the field being supposed at zero potential. Hence, denoting the numeric of the capacity of a given conductor by C, those of its charge and potential by Q and V respectively, we have $C = QV^{-1}$, and for $[C]$ therefore $[QV^{-1}]$, that is $[L]$. The unit of capacity has therefore the same dimensions as the unit of length; and the capacity of a conductor is properly expressed in C.G.S. electrostatic units as so many centimetres. Capacity.

Specific Inductive Capacity.

Specific Inductive Capacity $[K]$. The specific inductive capacity of a dielectric is (Chap. I., Section V.) the ratio of the capacity of a condenser, the space between the plates of which is filled with the dielectric, to the capacity of a precisely similar condenser with vacuum as dielectric; or, according to Maxwell's Theory of Electric Displacement (p. 33), it is defined as the ratio of the electric displacement produced in the dielectric to the electric displacement produced in vacuum by the same electromotive force. It is therefore in the electrostatic system simply a numerical coefficient which does not change with the units. Hence $[K] = 1$.

Current.

Electric Current $[\gamma]$. An electric current in a conducting wire is measured by the quantity which passes across a given cross-section per unit of time. If q be the numeric of the quantity which has passed in a time of which the numeric is T, then denoting the numeric of the current by γ, we have $\gamma = q/T$, and $[\gamma]$ is $[qT^{-1}]$ or $[M^{\frac{1}{2}}L^{\frac{3}{2}}T^{-2}]$.

Resistance.

Resistance $[r]$. By Ohm's law the resistance of a conductor is expressed by the ratio of the numeric r of the difference of potentials between its extremities to the numeric γ of the current flowing through it. We have therefore $r = v/\gamma$, and $[r]$ is $[v\gamma^{-1}]$ or $[L^{-1}T]$.

Conductivity.

Conductivity.* The change-ratio of conductivity is plainly $[LT^{-1}]$. The change-ratio for conductivity in electrostatic measure is thus the same as that for

* Mr. Oliver Heaviside has proposed to use the term *Conductance*, in the sense here given to *Conductivity*, of the reciprocal of a resistance. If this term, as seems desirable, be adopted, the term Conductivity might be appropriately reserved for what has been called *Specific Conductivity* (Chap. V. below).

velocity. Hence a conductivity in electrostatic C.G.S. units is properly expressed in centimetres per second.

Illustration of Conductivity as Velocity.

The following illustration of this result has been given by Sir William Thomson. Suppose a spherical conductor charged to a potential v to be connected to the earth by a long thin wire, of which the capacity may be neglected; and let r be the resistance of this wire in electrostatic measure. The current in the wire at the instant of contact is v/r. Now let the sphere diminish in radius at such a constant rate that the potential remains v. The current remains v/r, and the quantity of electricity which flows out in t seconds will be vt/r. If the radius be initially x, and in t seconds has diminished to x', the diminution of capacity is $x - x'$. Hence the loss of charge is $v(x - x')$, and we get $vt/r = v(x - x')$, or $1/r = (x - x')/t$. But $(x - x')/t$ is the velocity with which the radius of the sphere diminishes. The conductivity $1/r$ of the wire is therefore measured numerically by the velocity with which the surface of the sphere must approach the centre, in order that its potential may remain constant when the surface is connected to the earth through the wire.

CHAPTER IV.

GENERAL PHYSICAL MEASUREMENTS.

SECTION I.

MEASUREMENT OF ANGULAR DEFLECTIONS.

IT will save digressions and interruptions in what follows to give here some account of measurements which, although not themselves of an electric or magnetic nature, have constantly to be made in all electric or magnetic observations. The most important of these are: (1) The Measurement of Angular Deflections; (2) Measurements of Oscillations, including Determinations of Period, Amplitude, and Rate of Diminution of Amplitude of Vibrations; (3) Determinations of Couples and Moments of Inertia. There are other processes, such as weighing and the measurement and comparison of lengths; but these are described more fully than are the others in treatises on general physical manipulation, and are supposed to be known to a greater or less degree to the experimental student. Special methods or precautions necessary in particular cases will be indicated as they occur.

Measurement of Angles.

Angles of deflection are measured by the displacement of some form of index turning with the body deflected, and showing the magnitude of the deflection

on a properly arranged and fixed graduated scale. In the simplest arrangement the index is a thin material rod turning round an axis, and showing the deflections on a graduated circular arc, the centre of which is as nearly as possible in the axis. The scale is graduated to degrees, or, if it is of large radius, to some aliquot part of a degree as smallest scale division. The initial and deflected positions of the index relatively to the scale are read off, and their difference gives the deflection.

Ordinary Index and Scale.

For convenience of adjustment and accuracy in reading, the scale should be a complete circle, and the index extend across that circle, so that readings may be taken of the positions of both ends. The instrument should first be tested to see that the centre of the circular scale is accurately in the axis of rotation, and that the axis is at right angles to the plane of the circle, and in plane with and perpendicular to the index. The index is generally set at right angles to the axis with sufficient accuracy by the instrument-maker, and the adjustment in this respect can be tested by observing whether the index when turning round the axis remains in one plane. If readings are not to be taken with both ends of the index, it is not necessary that the index should be accurately at right angles to the axis. The point of the index in that case should turn in the plane of the scale. The axis can generally be set accurately at right angles to the plane of the circle, by changing the level of the apparatus. The index generally terminates in two sharp points, or bears two fine marks at its ends by which the readings are taken.

Adjustments.

Adjustments.

We shall call these the extremities of the index. The further adjustment consists in placing the extremities of the index and the axis in one plane, and causing the axis to pass accurately through the centre of the circle. Supposing the adjustment to have been approximately made, the index is made to play round the graduated circle, and the pairs of points on the scale which mark the positions of the extremities of the index for different deflections are carefully noted. When each line joining a pair of points passes through the centre of the circle the adjustments have been properly made. If the lines all pass through one point which is not the centre, the axis is in plane with the extremities, but does not pass through the centre; if the lines are all at the same perpendicular distance from the centre, the axis passes through the centre, but the extremities of the index are not in plane with the centre.

Indexes in electrical instruments are frequently thin glass tubes filled with some dark opaque substance; but an excellent index, thin, rigid, and light, is furnished by a fine tube of aluminium. This index can be handled and adjusted with ease, and its use avoids the danger of breakage which exists in the case of glass fibres.

Observation of Deflections.

In considering how the adjustments are to be tested, we have supposed that the deflections can be accurately observed; and the usefulness of the apparatus for exact measurements depends on the means provided for that purpose. In many instruments the index is so long and so mounted that its extremities project over the divisions of the scale, and the deflections are simply read by the observer looking as nearly as he can normally

at the plane of the circle. Unless, however, the index is very close to the scale, this process introduces a liability to error from parallax, that is, different readings are obtained according to the direction in which the scale is viewed. To prevent this source of error, the scale is frequently engraved on silvered glass, or is surrounded by a slip of silvered glass, in which the extremities of the index are seen by reflection, and the readings are taken always when the scale is so viewed that the extremities and their image in the silvered surface seem in coincidence.

Avoidance of Parallax.

For many purposes sufficient accuracy can be obtained by making each extremity of the index carry a small

Movable Vernier.

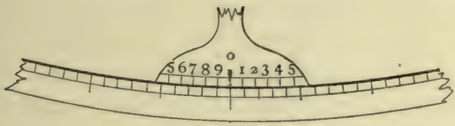

FIG. 37.

vernier. In the case of an aluminium index the ends can be two horizontal flat pieces on which the vernier is engraved. The middle line of division of the vernier is the zero (which contains an even number of divisions, as, for example, 10 corresponding to 9 of the scale), and the divisions are marked so as to read from 0 to 5 on the right, and from 5 to 10 (which coincides with zero) on the left, as shown in the diagram. The vernier carries at one end a projecting point, the image of which is seen in the silvered glass on which the scale is engraved,

Movable Vernier.

or in a piece of silvered glass surrounding the scale, and thus parallax is avoided.

In some cases parallax is avoided by having the scale circle in relief, so that the index can be made to move nearly in the plane of the scale, and with its extremities close to it. The positions of the extremities may be read either with or without a vernier. If a finely-divided vernier and scale are used, the readings may be taken by a microscope or a magnifying glass, since both graduations are in focus at once.

Limits of Accuracy Obtainable.

In some instruments, as in the Dip Circle, the index is a metal arm moving round a finely-divided circle, and carrying at its extremities verniers, the readings of which can be obtained by microscopes. The amount of accuracy obtainable here, all other adjustments being supposed correctly made, depends upon the fineness of the graduation and the precision with which the zero of the vernier can be placed in the desired position. Thus, for a vernier and scale which can be read to say 10″ of angle, the error of estimation of position must be less than 10″ or the fineness of the graduation is not taken advantage of. The reading microscope must of course be of sufficient power to take full advantage of any given degree of fineness of graduation.

Mirror Method.

Angular deflections are, however, generally measured in ordinary electric measurements by what is called the mirror method. A plane mirror or a concave spherical mirror is mounted so that the axis round which the rotation takes place is in the reflecting surface of the mirror if that is plane, or is a tangent to the reflecting surface

at its centre if the mirror is concave. The angle of rotation is measured by observing the angular distance between the positions of the image of a luminous object placed in front of the mirror. The index here is a ray of reflected light.

Ordinary Projection Arrangements with Concave Mirror.

In the ordinary or projection method this object is a short narrow slit, with its length parallel to the axis, made in an opaque diaphragm in front of a source of light, or, better, a somewhat large illuminated opening in the diaphragm, with a thin opaque wire or hair

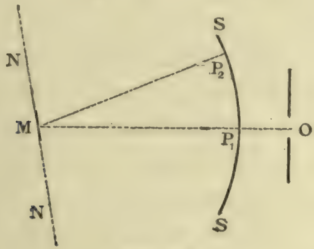

FIG. 38.

stretched across it parallel to the axis. We shall suppose first that the mirror is concave. The image of this slit or wire is received on a screen placed in the focal plane conjugate to that of the slit or wire. On this screen is ruled a scale, generally to half-millimetre divisions, by which the deflections can be measured. For considerable deflections the screen may be a graduated scale, with its length at right angles to the axis of rotation on a concave cylindric surface, the axis of which coincides with that of rotation. For example, Fig. 38 shows a plan of such an arrangement in a plane

P 2

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Arrangements with Concave Mirror.

perpendicular to the axis. O is the opening bisected by the opaque wire, M the mirror in its deflected position, shown by the line NN, P_1, P_2 the positions of the image on the scale SS corresponding to the undisturbed and deflected positions of the mirror. The illuminated opening and the scale are placed on opposite sides of a plane normal to the axis through the centre of the mirror, in order that the scale may not intercept the light. The method has the drawback that it is generally necessary to darken the scale, so that the illumination produced by the reflected ray may be distinctly visible. This is accomplished sometimes by placing the whole apparatus in an alcove with curtains, and sometimes by turning the back of the scale to the general light of the room, and shading the front above and at the ends with a projecting hood of dull black material.

Choice of Scales.

Well graduated paper scales are easily obtained, and when properly mounted, and *afterwards* compared with a standard scale are quite reliable. They should be well glued to a backing of hard thoroughly seasoned wood, and the rest of the wood and the scale itself well covered with spirit varnish to prevent the absorption of moisture. When it is convenient to use a straight scale, one graduated on glass or ivory is preferable, and the former can be easily made by the experimenter himself by copying the graduation from a standard scale. Scales graduated on glass roughened so as to be semi-opaque are very convenient, as the image of the hair or wire and the divisions can be seen, and the deflection read from behind the scale.

If accurate readings for large deflections are required,

the scale cannot be made circular, but must be so curved that, for the distances P_1M, OM chosen, a distinct image of the wire may be formed on the scale for all deflections. The graduation of the scale, if previously made in equal divisions, must then be compared with a circular scale, in order that the deflections may be reduced to angle.

Reduction of Reflection to Radian Measure.

If n be the number of equal degree divisions which measures the distance P_1P_2 on the scale, if that is circular, or the corresponding number on a circular scale of radius MP_1, r the distance MP_1, also in scale divisions, θ the deflection of the mirror in radian measure, then since $P_1MP_2 = 2\theta = n/r$, we have

$$\theta = \frac{n}{2r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In an arrangement commonly used the slit or wire and the scale are nearly at the same distance from the mirror. The distance of the scale is then twice the focal distance of the mirror.

Arrangement of Plane Mirror with Lens.

Instead of a concave mirror an equivalent arrangement is sometimes adopted, consisting of a plane mirror with a convergent lens placed close in front of it. With the reversal of the ray by reflection, O and P_1 are now conjugate foci of a system of lenses each identical with that used, having a distance between their optical centres equal to twice the distance of the optical centre of the lens from the mirror. The arrangement may be made conveniently in some cases by silvering the plane face of a plano-convex lens, and is then optically equivalent to two similar plano-convex lenses placed with their plane faces in contact.

Use of straight Scale.

A straight scale placed at right angles to the axis, round which the mirror turns, is frequently used. Let α be the angle which the reflected ray in the undisturbed position of the mirror makes with a plane at right angles to the scale and passing through the centre of the mirror, n the corresponding distance on the scale, α' and n' the corresponding quantities for a deflected position, r the perpendicular distance of the scale from the mirror, θ the angular deflection of the mirror, then

$$\tan 2\theta = \frac{\tan \alpha' - \tan \alpha}{1 + \tan \alpha \tan \alpha'} = \frac{n' - n}{r^2 + nn'}\, r$$

and

$$\theta = \frac{1}{2} \tan^{-1} \frac{n' - n}{r^2 - nn}\, r \quad . \quad . \quad . \quad . \quad (2)$$

Deduction of angular Deflection from Readings.

If n is zero we have

$$\theta = \frac{1}{2} \tan^{-1} \frac{n'}{r} \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If n' be small in comparison with r we have approximately from (2)

$$\theta = \frac{1}{2} \frac{n' - n}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and from (3)

$$\theta = \frac{1}{2} \frac{n'}{r} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Telescope Method.

For more exact measurements when the deflections are small, Poggendorff's arrangement of telescope and scale is used with a plane mirror. The telescope has a positive eyepiece with two mutually rectangular intersecting cross-wires at its focus. Its line of collimation *

* The line joining the optical centre of the object glass with the intersection of the cross-wires.

is arranged to be nearly in plane with and at right angles to the axis. In most cases in which the telescope is used the axis is vertical, and we will for definiteness assume that this is the case. One of the cross-wires is then in a vertical, the other in a horizontal, plane. The scale is straight and the numbers on it are engraved in such a manner that they appear erect, and are read from left to right when seen through the telescope. Telescope Method.

The scale must be well illuminated, and the method has the advantage that in general this is sufficiently accomplished for an opaque scale if it is placed in a

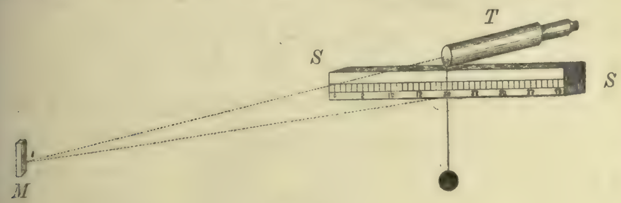

FIG. 39.

well-lighted room; and the local darkening required in the ordinary projection mirror method is thus avoided. Glass scales, constructed either by graduation on the surface of transparent glass, or on a silvered surface, may be used with convenience. The scales are illuminated by light from behind; and in the former case the divisions appear dark on a bright ground, in the latter shine out brilliantly on a dark ground. The room plainly must be darkened if such scales are used.

The arrangement of the apparatus is shown in the sketch, Fig. 39. T is the telescope, SS the scale, M the

Adjustments in Telescope Method.

mirror. The telescope as a whole is movable for adjustment both in a horizontal and in a vertical plane. The various parts are set up and adjusted as follows. The telescope eye-piece is adjusted so that distinct vision of the cross-wires is obtained. The eye-piece as a whole is then moved until an object, at a distance equal to twice that at which the telescope is to be placed from the mirror, is distinctly seen without parallax at the cross-wires. A plummet is hung below the object-glass by a fine wire in a vertical plane through the centre of the object-glass. The telescope is then placed in front of the mirror, supposed at rest in the undisturbed position, and is moved about until the plummet wire is seen by reflection in the mirror. The scale, which is usually supported by an adjustable holder carried by the telescope stand, is placed in position below the object-glass, so that the plummet wire is seen in the telescope coincident with the middle division of the scale. The scale is levelled, and adjusted by direct measurement, so that any two points at equal distances on opposite sides of the middle division are at equal distances from the centre of the mirror. The final adjustment of the telescope for parallax and distinct vision is now made.

In order that any accidental disturbance of the telescope may be rectified with certainty, a fixed mirror is set up close to the movable one, and the part of the scale seen at the intersection of cross-wires by reflection from this mirror is read off and recorded.

If the adjustments have been properly made, the divisions on the scale are now seen clearly in the mirror; and this distinctness is practically the same for all parts

of a scale of ordinary length placed at the usual distance from the mirror. A deflection of the mirror causes the divisions of the scale to pass across the field of view of the telescope; and if the mirror comes to rest in a new position, the angle of deflection can be obtained by comparing with the initial reading the new reading, which coincides with the vertical cross-wire. Let n be the deflection measured along the scale in terms of a scale division as unit, r the distance of the scale from the mirror in terms of the same unit, θ the angle of deflection of the mirror in radian measure; then as in (3)

Deduction of Angle of Deflection from Readings.

$$\theta = \frac{1}{2}\tan^{-1}\frac{n}{r}.$$

If the axis round which the mirror turns is not in the reflecting surface, but at a distance δ from it, we have plainly

$$\theta = \frac{1}{2}\tan^{-1}\frac{n}{r - \dfrac{\delta}{\cos\theta}} \quad . \quad . \quad . \quad . \quad (6)$$

Since δ is in general small, this may be calculated with sufficient accuracy by finding θ from the previous formula, and using its value in the calculation of the corrected value from the right-hand side of (6). If, however, θ be very small, $\cos\theta$ on the right may be taken as unity, and we have

$$\theta = \frac{1}{2}\,\frac{n}{r - \delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In order that a point in the middle of the portion of the scale seen in the telescope may have the greatest

Relation of Breadth of Mirror to Diameter of Object Glass.

possible illumination, it is necessary that the mirror should be at least so broad that for the positions of telescope and scale adopted the pencil of rays from the point should fill the horizontal diameter of the object-glass. Hence if r, r' be the distances of the scale and telescope respectively from the mirror, δ the diameter of the object-glass, the minimum breadth of the mirror is $r\delta/(r + r')$.

Length of Scale visible.

The portion of the scale visible in the telescope may be defined as that from no point of which the part of the pencil received by the eye-glass fills less than a certain fraction of the diameter of the object-glass. If we define it by half the diameter we get when the breadth of the mirror is b, (if the real optical extent of field of view of the telescope be not exceeded) for the length visible $b(r + r')/r'$. The pencil received from either extremity of this space covers the object-glass as far as its centre. With the above minimum breadth of mirror, the extent of the field of view is $r\delta/r'$, which is greater the smaller is r' in comparison with r. If, as usually is the case, $r = r'$ nearly, it is simply equal to the diameter of the object-glass.

Limits of Accuracy obtainable.

The precision of the readings with this arrangement depends upon the angle of deflection which corresponds to the smallest visual angle visible in the telescope. Calling this angle η, the corresponding length of scale is $\eta(r + r')$, and the angle which this subtends at the mirror is $\eta(r + r')/r$. The smallest angle of turning of the mirror visible is therefore $\eta(r + r')/2r$. The smaller r' is in comparison with r the smaller is this angle; hence the sensibility is increased by placing the scale at

a greater distance from the mirror than the telescope. The minimum breadth of mirror becomes as r' is diminished more and more nearly equal to the diameter of the object-glass; hence if a mirror of sufficient size is available, and it is convenient to construct and well illuminate a long scale, the distance of the scale may with advantage be made considerably greater than that of the telescope.

Telescope with Concave Mirror, or with Plane Mirror and Lens.

A telescope may be used with a concave mirror (or the equivalent arrangement of plane mirror and convergent lens placed close to it) and curved scale as described above. All that is necessary is to substitute the telescope for the slit or illuminated wire, and arrange it for distinct vision of the scale without parallax at the cross-wires. The best arrangement, however, is now to place the scale at a distance from the mirror equal to the principal focal distance of the mirror, or the corresponding distance in the case of the lens and plane mirror. If the radius of curvature of the mirror be R, this distance is $\frac{1}{2}R$. For the lens, if f be its principal focal distance and δ the distance of its optical centre from the mirror, the distance is by the theory of lenses $f(f - 2\delta)/2(f - \delta) + \delta = \frac{1}{2}(f + \delta) - \delta^2/2(f - \delta)$. Hence if δ is small in comparison with f, the distance is approximately $\frac{1}{2}(f + \delta)$. If the lens be plano-convex and silvered on its plane face, δ is zero and the distance is $\frac{1}{2}f$.

Visible Length of Scale.

Since the rays of light received by the telescope are parallel, the telescope has only to be adjusted for distinct vision of a distant object, and may be placed at any distance from the mirror, with no other alteration

than a change in the extent of scale seen in the field of view. For a spherical mirror of radius R, the length of scale visible with a mirror of breadth b in a telescope with object-glass at distance r' is plainly $bR/2r'$. In the case of the lens, if b be its diameter, the length of scale visible is practically bf/r'.

Relation of Mirror Diameter to Diameter of Object Glass.

Plainly in order that the whole power of the telescope may be utilised, the parallel beam from the mirror or from the lens due to a point in the middle of the portion of the scale visible must fill the object-glass, that is, the mirror or lens must be at least equal in diameter to the object-glass. The smallest angular deflection observable is then obviously half the smallest visual angle observable in the telescope, that is, half its space-penetrating power. This power varies directly as the diameter of the object glass, and it has been found that an object glass of 15 centimetres diameter is necessary to separate two objects which subtend an angle of 1″ at its centre.

Smallest Visible Deflection.

SECTION II.

MEASUREMENTS OF OSCILLATIONS.

Utility of Observation of Oscillations.

The most important species of oscillations for us to consider are oscillations of a body round an axis, such oscillations for example as are performed by a body suspended by an elastic wire under the influence of torsion, or the vibrations of a magnet in a magnetic field, with or without the damping action of induced

currents, or of frictional resistance proportional to the velocity. A determination of the period and rate of subsidence of the oscillations, with a knowledge of the moment of inertia of the vibrating body round the axis of rotation, gives a means of calculating the moving and resisting forces acting on the body.

Equation of Motion for Simple Harmonic Oscillations.

For simple harmonic oscillations of diminishing range the equation of motion is

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{L}{\mu}\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where θ is the angular deflection at time t. For the criterion of simple harmonic motion of constant range is that the body should be acted on by a system of forces or couples proportional to the displacement and acting towards the equilibrium position. The moment of the couple-system in the present case is $L\theta$. Besides this system of couples we suppose a retarding system of forces to act on the body with a moment round the axis proportional at every instant to the angular velocity, and equal to $2\mu k d\theta/dt$, where μ is the moment of inertia of the moving system round the axis, and k a constant. Equating the rate of diminution of moment of momentum $-\mu d^2\theta/dt^2$ to the sum of moments $2\mu k d\theta/dt + L\theta$ we get the equation of motion (8).

Exponential Solution.

The general solution of (8) is, writing n^2 for L/μ,

$$\theta = A_1\epsilon^{m_1 t} + A_2\epsilon^{m_2 t} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where A_1, A_2 are constants, and m_1, m_2 are the roots of the quadratic equation

$$m^2 + 2km + n^2 = 0,$$

Realised Solution for Imaginary Roots of Auxiliary Quadratic or Simple Harmonic Solution.

that is, $m_1 = -k + \sqrt{k^2 - n^2}$, $m_2 = -k - \sqrt{k^2 - n^2}$. These roots are imaginary if $n^2 > k^2$, and in this case the solution takes the form

$$\theta = \epsilon^{-kt}(C\cos\sqrt{n^2 - k^2}t + C'\sin\sqrt{n^2 - k^2}t). \quad (10)$$

If t be reckoned from the instant of greatest elongation Θ in the positive direction (10) becomes

$$\theta = \epsilon^{-kt}\Theta\sin\left\{(n^2 - k^2)^{\frac{1}{2}}t + \frac{\pi}{2}\right\} \quad . \quad . \quad (11)$$

The motion is therefore simple harmonic with period $2\pi/(n^2 - k^2)^{\frac{1}{2}}$, and amplitude diminishing at logarithmic rate k.

Non-oscillatory Subsidence.

If $n^2 < k^2$, m_1, m_2, are real and the motion is non-oscillatory; that is, any displacement given to the system disappears by a motion of the body towards the position of equilibrium without oscillation about that position. This condition is fulfilled in aperiodic or "dead beat" instruments.

Unresisted Oscillation.

When k is so small that the resistance is negligible in comparison with the other forces, the equation of motion (8) becomes

$$\frac{d^2\theta}{dt^2} + n^2\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and the solution corresponding to (11)

$$\theta = \Theta \sin\left(nt + \frac{\pi}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Case of Forces varying as Sine of Displacement. Equation of Motion.

A case which frequently occurs is that in which the motion is oscillatory but not simple harmonic, owing to the fact that the moment of the system of couples is proportional, not to θ, but to $\sin\theta$. Assuming zero or negligible resistance, the equation of motion is—

$$\frac{d^2\theta}{dt^2} + n^2 \sin\theta = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where n, as before, denotes L/μ. This is the case of which a circular pendulum vibrating through a finite arc is the type. Denoting the amplitude in radian measure by a, we have, multiplying by $d\theta/dt$ and integrating from $\theta = a$ to $\theta = 0$,

$$\left(\frac{d\theta}{dt}\right)^2 = 2n^2(\cos\theta - \cos a).$$

Hence

$$dt = \frac{1}{\sqrt{2}n}(\cos\theta - \cos a)^{-\frac{1}{2}}d\theta \quad . \quad . \quad . \quad . \quad (15)$$

Calculation of Period.

and the integral of this, taken between the limits 0 and a, is the quarter period $= T/4$.

Let

$$\sin\frac{\theta}{2} = \sin\frac{a}{2}\sin\phi,$$

then (15) becomes

$$dt = \frac{1}{n}(1 - \sin^2\frac{a}{2}\sin{}^2\phi)^{-\frac{1}{2}}d\phi \quad . \quad . \quad . \quad . \quad (16)$$

Expanding and integrating the series term by term between the limits 0 and $\frac{1}{2}\pi$ (which correspond to 0 and a), we get—

$$T = \frac{2\pi}{n}\left\{1 + \left(\frac{1}{2}\right)^2 \sin^2\frac{a}{2} + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \sin^4\frac{a}{2} + \&c.\right\} \quad . \quad (17)$$

Deduction of Period for Infinitely small Range.

The first term of this series $2\pi/n$ is the ordinary simple pendulum formula for an infinitely small amplitude of vibration. To find from values of T derived from observation of oscillations of this kind the corresponding period for infinitely small oscillations, it is only necessary to divide the value of T thus found by the value of the series within the brackets. For almost all practical purposes it is sufficient to use the approximate equation—

$$T = \frac{2\pi}{n}\left(1 + \frac{a^2}{16}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which, for an amplitude of half a radian (28° 42′ nearly), is true to 1/50 per cent., and to a higher degree of accuracy for smaller amplitudes.

A table of reducing factors calculated from (17) is given at the end of this volume.

The period of an oscillation is the time-interval between two successive passages of the moving body through the same position in the same direction. For the determination of this a means of observing the position of the body and a time measurer are necessary. For many physical purposes a fairly accurate clock or chronometer beating audibly seconds or half seconds, or a good watch is sufficient. In well-equipped laboratories, chronographs are available, by which, at the successive instants of occurrence of the phenomenon observed, marks can be made on a ribbon of paper or revolving drum kept moving uniformly (checked, of course, by a break-contact chronometer or pendulum, which breaks an electric circuit at equal intervals of time, and makes a mark on the ribbon at each break)

by clockwork at a known rate recorded by the instrument itself. We shall suppose that an audibly ticking chronometer is used, and it will be easy to modify the methods of observation to suit a registering time-measurer.

Method of Telescope and Mirror.

For observation of the vibrating body, when it is only necessary to find the instant of passage through a given position, it is convenient to use a telescope focussed so as to give an image of some part of the body at the intersection of cross-wires when the body is in the position in question. The best position is that of the vibrating body when at rest with no forces acting upon it. A vertical line is drawn upon the body, and focussed at the cross-wires in the usual way by first adjusting the eye-lens and cross-wires for distinct vision of the latter, and then moving the whole eye-piece until distinct vision of the mark is obtained.

It is, however, more generally convenient to mount a light mirror on the body and use the arrangement of telescope and long circular scale described above. The apparatus is adjusted so that the division of the scale in the same vertical plane as the centre of the object-glass and the centre of the mirror is at the intersection of cross-wires when the body is at rest. Or, and preferably, where the equilibrium position of the vibrating body is liable to change, the exact reading of the scale corresponding to the equilibrium position, or *zero-reading*, as we shall call it, may be obtained as follows, while the body is vibrating. The readings of the scale when the body is at rest at three consecutive times are taken. Let n_1, n_2, n_3 be the readings, a the zero-reading, T the

Determination of "Zero Reading."

Observation of Transit of Zero.

period (not necessarily known), then for the deflection from the zero in the first case we have $d = n_1 - a$, and in the second and third cases by (11) above, $n_2 - a = - \epsilon^{-kT/2}d$, $n_3 - a = \epsilon^{-kT}d$. Hence we get

$$a = \frac{n_2{}^2 - n_1 n_3}{2n_2 - n_1 - n_3} \quad . \quad . \quad . \quad . \quad (19)$$

When n_1 does not differ greatly from n_3 this equation becomes

$$a = \tfrac{1}{4}(n_1 + 2n_2 + n_3) \quad . \quad . \quad . \quad . \quad (20)$$

If the rate of diminution of amplitude is not great, the extreme readings of a greater odd number of successive semi-vibrations may be read off. The arithmetic means of the readings taken at each side are found separately, and the arithmetic mean of the two results is the required reading for the middle position. Thus let d_1, d_2, d_3, d_4, d_5 be readings of the scale for five successive elongations (points of extreme deflection), so that d_1, d_3, d_5 are the extreme readings for deflection to the left, d_2, d_4 for deflection to the right. Then $(d_1 + d_3 + d_5)/3$ is the mean extreme reading on the left. Similarly $(d_2 + d_4)/2$ is the mean extreme reading on the right. The zero reading is then approximately the mean of these, or $(d_1 + d_3 + d_5)/6 + (d_2 + d_4)/4$. It is necessary thus to take one more reading on one side than on the other in order that in the case of diminishing amplitude the mean for one side may correspond to that for the other. Thus if three readings, d_2, d_4, d_6, were taken on the right, the mean $d_2 + d_4 + d_6$ would be the mean of readings taken one by one later than

those on the other side, and their mean would therefore be too small to give accurately the zero reading.

Observations of Period. The instant at which the zero reading of the scale makes its transit is now observed while the body is vibrating. This is done as follows:—The observer takes time, say at the end of a minute, and then listening to the ticking of the clock, counts on from the end of the minute until the middle reading passes the intersection of cross-wires. The division of the scale at the intersection of cross-wires at the beat of the chronometer before and at the beat after the instant are, if possible, also observed. From these the exact instant of the transit of the zero reading can be found by assuming the velocity of transit constant between the two beats. If both these readings cannot be obtained, that one nearest in time to the transit of the zero reading is read off, and from the approximately known velocity of transit the interval between the beat and the passage can be found. The observer allows the vibration to continue, and counts the transits past the cross-wires until some convenient number, say 10 or 20, have taken place. Before the end of the series, glancing at the clock, he takes time, and then counts the ticks until the last transit of the series has taken place, and makes the same observations as at the beginning of the series. He repeats these observations for successive series, and thus obtains an approximate measurement of the time-interval from the first to the last transit of each series.

From this time-interval for one or more series the period of oscillation can be approximately calculated.

The mean of the periods calculated from each set of observations will give the average period with accidental errors of a particular set more nearly eliminated.

Each set of determinations of period ought to be immediately preceded and followed by observations of amplitude. For this it is only necessary to observe three elongations or scale-readings at the stationary points. Thus let three successive scale readings be d_1, d_2, d_3, the total range of apparent motion of the scale across the field of view is $d_2 - \frac{1}{2}(d_1 + d_3)$, or the amplitude of oscillation reckoned on a circle of radius equal to that of the scale is $\frac{1}{2}d_2 - \frac{1}{4}(d_1 + d_3)$. If a denote the amplitude in radian measure, and r the radius of the scale in degree divisions, then

Observations of Amplitude.

$$a = \frac{\frac{1}{2}d_2 - \frac{1}{4}(d_1 + d_3)}{r} \quad . \quad . \quad . \quad . \quad (21)$$

The mean of the two values of the amplitude obtained at the beginning and end of each series of observations may, if the series does not extend over too long a time, be taken as the amplitude during the whole series, and used, if the oscillations are of the kind which require it, for the reduction of the period obtained from the series to that which would have been obtained if the amplitude had been infinitely small (see p. 223, above). The mean value of the periods obtained after all corrections from the various series of observations may be taken as the period required.

In cases in which the rate of diminution of amplitude is small, a large number of oscillations may be made in a series extending over a considerable time. A first

Observation of a Long Series of Vibrations.

short series of observations will suffice to give an approximate value of the period, and this can be used to save the necessity of counting the number of oscillations in the long series. The times t, t' of the beginning and end of the long series (with, as before, the amplitude before and after) are observed and recorded. Then if T' be the rough value of the period before obtained, which it is known may have an error not greater than τ, we have for N the number of vibrations in the series

$$N = \frac{t' - t}{T'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

with a possible error of $\pm N\tau/T$. If τ is so small that $N\tau$ is less than half a period, then plainly N is the actual number of vibrations made during the interval, and the true period T can be at once obtained.

Definition of Logarithmic Decrement.

When the resistance to motion is proportional to the velocity, each amplitude has a constant ratio to the succeeding amplitude, and this ratio is called the *Logarithmic Decrement* of the motion.

Its Determination.

It is obtained at once from observations of amplitude. The zero reading having been determined, an odd number of successive elongations on the same side of zero are observed, and the arithmetic mean of the deflections from zero taken. The result is the amplitude for the middle vibration of the series. The vibratory motion is allowed to continue undisturbed for some time and another series of observations then made. Then if we call the semi-period for which the first amplitude was determined the m^{th}, and the second semi-period to which the second amplitude corresponds

be the $(m+n)^{th}$, the time between the two observations is $n-1$ semi-periods. Hence if λ be the logarithmic decrement, Θ_m the first amplitude, Θ_{m+n} the second, then

Its Determination.

$$\Theta_m = \lambda^{n-1}\,\Theta_{m+n}$$

or

$$\lambda = \frac{1}{n-1}\log\frac{\Theta_n}{\Theta_{m+n}} \quad . \quad . \quad . \quad . \quad (23)$$

If $2k$ denote, as in (8) above, the ratio of the moment of the resistance to the product of the moment of inertia and the angular velocity, or, which is the same thing, the ratio of the angular retardation to the angular velocity, and T one period, then we see by (11) that

$$\lambda = kT. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Period, &c. deduced from Successive Series of Observations.

When the oscillations are long continued, a short series of observations of an odd number of successive elongations, with the time of zero passage for each semi-vibration, is made from time to time. The number of oscillations which has taken place between every two successive series is determined as described in p. 228, and the results combined as follows. The interval between the time of zero passage in the first semi-vibration in the first series and the zero passage in the first of the second series divided by the number of semi-periods gives an average semi-period of vibration; in the same way the second semi-vibration of the first series and the second of the second series gives another average, and so on. A second set of averages can be

Period, &c. deduced from Successive Series of Observations.

obtained from the second and third short series and the interval between them.

Each successive set of average results (corrected if necessary for amplitude) gives a mean result, and these again a final mean period. In the same way the amplitudes may be dealt with and a mean logarithmic decrement found.

The results of observations, and of different sets of observations, may be combined by rules derived from the theory of *Least Squares* so as to give the most probable values of the quantities sought, and we shall state and use such rules when necessary in what follows. We have not space here to enter into the subject.

Section III.

MEASUREMENT OF COUPLES.

There are many electric and magnetic instruments in which an applied couple or system of couples is equilibrated by a reacting couple the moment of which therefore measures that of the former. This equilibrating couple may be obtained in various ways; for example, by the torsion of an elastic wire or thread; by means of a bifilar suspension; or by the deflection of a magnet in a magnetic field. We shall consider shortly the measurement of couples produced in the first two different ways, deferring the third until we deal with magnetic measurements.

Definition of Amount of Twist.

When a wire is twisted by turning one end round the axis of the wire while the other end is held fixed, a resisting couple is set up by elastic reaction in every cross-section where twist has been given. The moment of this couple at any one section is proportional to the *twist* there existing defined as follows:—Let $\delta\theta$ be the angle through which one of two normal cross-sections at a distance δl apart along the axis of the wire has been turned relatively to the other, then $\delta\theta/\delta l$ is called the *average twist* over that portion of the wire. Let the middle cross-section of this portion of the wire be that at which the twist is to be measured, and let δl be made very small, $\delta\theta/\delta l$ is then the *twist* required.

Torsional Rigidity defined.

Along a straight homogeneous wire of uniform cross-section, attached at its extremities to two bodies by the relative motion of which round the axis of the wire the twist is applied, the distribution of twist, except near the fastenings at the ends, is uniform. The couple of elastic reaction in a particular wire of circular cross-section and isotropic material is equal to the product of the twist, the *rigidity* (defined below) of the material, and the moment of inertia round the axis of a circular disc of diameter equal to that of the wire, and of unit mass per unit of area. The product of the last two factors is called the *Torsional Rigidity* or *Modulus of Torsion* of the wire, and is the elastic couple set up in the wire per unit of twist. This is the constant which for wires used as torsional suspensions it is necessary to determine.

Rigidity defined.

The simple rigidity of the material which forms one factor of the torsional rigidity is defined by supposing

Rigidity defined. equal tangential force per unit of area to be applied over four forces of a unit cube so that the forces in each face act as shown by the arrow-heads in Fig. 40. By the action of these forces the cube is distorted so that the section in the plane of the paper becomes a rhombus, that is, each of one pair of opposite angles becomes less than a right angle by a certain amount, and each of the other angles greater than a right angle by the same amount. The *rigidity* is the ratio of the tangential force per unit area thus applied, to the amount in

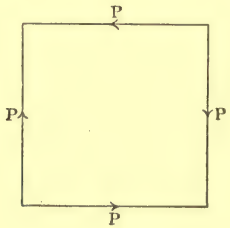

FIG. 40.

radian measure by which each angle of the section of the cube has been changed.

Determination of Torsional Rigidity. Method of Oscillations.

The torsional rigidity of a wire is best determined by the method of oscillation. The wire is suspended vertically from a fixed support which securely holds its upper end from turning, and to the lower is attached a body of such form that its moment of inertia round the axis of the wire can be readily determined from its mass and dimensions.

The best form of vibrator is for several reasons a

Form of Vibrator.

hollow circular cylinder of brass or copper suspended so that the axis is in line with that of the wire. In such a cylinder the thickness of the metal, in most practical cases, need not be so great in comparison with either radius that defects in homogeneity will seriously affect the correctness of the moment of inertia, as deduced from the mass and dimensions of the vibrator. Further, its form is one for which the motion is but slightly affected by the presence of the air in the chamber in which the oscillations take place.

The cylinder is suspended by means of a cross bar of the same material and of rectangular section which passes through apertures at the opposite extremities of a diameter of the cylinder near its upper edge. The length of each aperture is exactly equal to the breadth of the bar so as to avoid side-shake, but their depths may be slightly greater than the thickness of the bar. Symmetrically placed with reference to the centre of the bar are two notches in its upper surface which exactly fit the upper edges of the orifices, so that when the whole is suspended by the centre of the bar the cylinder rests in these notches with its plane horizontal, and all is symmetrical about the axis, and practically rigid for motions round it.

Coincident with the axis is a small hole in the bar, through which the wire can be passed, or in which a small vertical wire can be fixed to fit the clamp with which the lower end of the wire may be terminated.

If the object of the experiment is to determine the torsional rigidity of a particular wire used in an instru-

Form of Vibration.

ment of any kind, and this cannot be done with the wire in position, the wire should have a clamp or fastening at each end permanently fixed to it for securing it in any new position; and the total weight on the wire when vibrating should be as nearly as may be the same as that borne by it when at its proper use.

The period and rate of subsidence of the oscillations are observed and the results dealt with in the manner already described (pp. 225—230 above).

In the case of a cylindric vibrator the mirror may be dispensed with, and a scale of equal divisions engraved on or cemented round the upper or lower edge of the cylinder. This scale is viewed through a telescope directly, and as the wire vibrates the divisions pass across the field of view, so that the time of passage of any one division and the divisions of greatest elongation can be observed.

Deduction of Rigidity from Results.

If τ denote the torsional rigidity of the wire, n the rigidity of the material, and a the radius of the wire, then, as stated above, p. 231.

$$\tau = \frac{1}{2}\pi n a^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

If T denote the period, kT the logarithmic decrement, μ the moment of inertia of the whole vibrating system, a the radius, and l the length of the wire, n is given by the equation

$$n = \frac{2\mu l}{\pi a^4}\left(\frac{4\pi^2}{T^2} + k^2\right) \quad . \quad . \quad . \quad . \quad (26)$$

Deduction of Rigidity from Results.

or if V be put for the volume of the wire

$$n = \frac{2\pi\mu l^3}{V^2}\left(\frac{4\pi^2}{T^2} + k^2\right). \quad . \quad . \quad . \quad (27)$$

If k can be neglected (28) becomes

$$n = \frac{8\pi^3\mu l^3}{V^2 T^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

a form sometimes used.*

Since the rigidity modulus thus determined involves the fourth power of the radius, and it is difficult to obtain a truly cylindric wire, values of n obtained thus for wires must in general be taken as only roughly approximate. It is, however, generally possible to find with accuracy the torsional rigidity τ of the wire. This is given by the equation

$$\tau = \mu l\left(\frac{4\pi^2}{T^2} + k^2\right). \quad . \quad . \quad . \quad . \quad (29)$$

Theory of Method.

If θ denote the whole angle through which one end of the wire has been turned relatively to the other, and l the length of the wire, then the twist is θ/l. Now consider two radii in the section represented in the diagram which are inclined at an infinitely small angle $d\psi$. These will intercept between the circles of radii r and $r + dr$, a small square, provided $dr = rd\psi$. This in the unstrained wire may be regarded as one face of a small cube which has two faces in infinitely nearly parallel planes through the axis and the two radii, and two other faces tangential to the two cylinders of radius r and $r + dr$, and the remaining face opposite to the first in a cross-section at a distance equal to dr. By the strain the angles between the first and last-mentioned faces and those in the radial plane have been altered by the amount $r\theta/l$. Hence the opposite tangential forces required, as in Fig. 40, over the faces in the cross-sections and on the two faces in the radial planes is for each face $n.rd\psi.dr.r\theta/l$.

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* See *Encycl. Brit.*, Art. "Elasticity," by Sir William Thomson.

Theory of Method.

The moment of this round the axis is $nr^3drd\psi \cdot \theta/l$. Hence $2\pi n\theta/l \cdot \int_0^a r^3dr = \frac{1}{2}\pi na^4\theta/l$ is the total moment over the cross-section. But this, divided by the twist θ/l, is the torsional rigidity. Hence (25) above.

The integral just found is the total moment of the elastic forces in each cross-section producing angular acceleration of the whole moving system towards the position of equilibrium. Besides these forces, the system is acted on by forces of viscous resistance (that is resisting forces depending on the velocity), partly in the wire and partly between the moving system and the air; and these we shall assume as everywhere proportional to the velocity, and therefore to have a moment round the axis directly proportional to the angular velocity $d\theta/dt$.

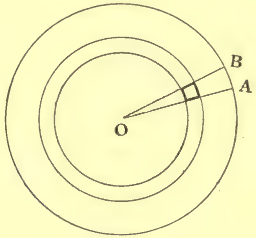

FIG. 41.

We shall denote this moment by $2k\mu d\theta/dt$. The equation of motion is then

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{1}{2}\frac{\pi na^4}{\mu l}\theta = 0 \quad . \quad . \quad . \quad . \quad (30)$$

the differential equation found on p. 221, above, which, under the conditions there stated, denotes simple harmonic oscillations with amplitude diminishing at logarithmic rate k. From the solution there given we have

$$T = \frac{2\pi}{\left(\frac{\pi na^4}{2\mu l} - k^2\right)^{\frac{1}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

which yields at once (26) and (27).

If the wire be not of truly circular section, τ, its torsional rigidity, must be used instead of $\pi\nu a^4/2$ in (30) and (31), which then give (29) above. Theory of Method.

Since the torsional rigidity of a cylindric wire is as the fourth power of its radius, or the square of its area of cross-section, it is frequently preferable to use a number of wires side by side to support a given load, rather than a single wire of equivalent cross-section. Thus such a wire equivalent in cross-section to n equal wires would not support a greater load, but would have approximately n times the torsional rigidity.

Moments of Inertia.

The moment of inertia of the wire is generally so small in comparison with that of the vibrator that it is negligible within the limits of the errors of observation. μ is therefore in general the result obtained from the known distribution of mass and the dimensions of the vibrator.

The moment of inertia round any axis is by definition the sum of the products of the mass of each small element of volume of the body into the square of its distance from the axis, and is therefore obtainable in general by integration. The results for the most useful cases are contained in the formula :—*

$$\left.\begin{array}{l}\text{Moment of inertia of}\\ \text{a uniform solid round}\\ \text{an axis of symmetry}\end{array}\right\} = \frac{\begin{array}{c}\text{Mass} \times \text{sum of squares of}\\ \text{rectangular semi-axes perp.}\\ \text{to axis of rotation}\end{array}}{3,\ 4, \text{ or } 5},$$

according as the solid is a rectangular parallelepiped, or thin elliptic plate (or solid elliptic cylinder if the

* Routh, *Rigid Dynamics*, vol. i. chap. i.

axis of the solid is the axis of rotation), or an ellipsoid.

Moment of Inertia of Cylindric Vibrator.

For the moment of inertia of a right circular cylinder of mass m and radius r, this formula gives $mr^2/2$: hence for that of a hollow cylindric vibrator of external radius r', internal radius r, and mass M, is $= M\,(r' + r'^2)/2$. For the bar by which it is suspended, supposed of uniform rectangular section, we have, if m be its mass, a its length, and b its breadth, $m(a^2 + b^2)/3$. From these two results the moment of inertia for the cylindric vibrator and attached bar can be found. The vibrator is weighed and its internal and external radii determined by careful measurement. The mass and length and breadth of the bar are also found as carefully as possible, and allowance made for the slots in which the cylinder rests.

Method by known difference of two unknown moments of Inertia.

It is possible to avoid the determination of the moments of inertia of the separate parts of a vibrator by using an arrangement of masses, the configuration of which can be changed so as to alter the moment of inertia without altering the mass and therefore also without affecting the pull on the wire.

The periods T_1, T_2 of vibration are observed for two such different configurations in which the moments of inertia and logarithmic decrements are μ_1, μ_2, k_1, k_2 respectively. Then we have from (27)

$$n = \frac{2l}{\pi a^4} \frac{4\pi^2(\mu_1 - \mu_2) + k_1^{\,2}\, T_1^{\,2}\mu_1 - k_2^{\,2}\, T_2^{\,2}\mu_2}{T_1^{\,2} - T_2^{\,2}} \quad . \; . \; (32)$$

This, if k_1, k_2 be small as is generally the case, depends on $\mu_1 - \mu_2$, which can with proper arrangement be determined with more ease than the values of μ_1, μ_2 separately.

A convenient arrangement for this purpose is Maxwell's Vibration Needle. A straight cylindric tube of brass contains four other tubes of brass, each exactly $\frac{1}{4}$ of the length of the outer tube so that they just fill up its length. Two of these inner tubes are empty, the other two are filled with lead. The vibrator made up of these tubes is hung horizontally from the wire by means of a straight rigid stem attached at right angles to the tube, in line with its centre of inertia. To the upper end of this stem the wire is clamped. Maxwell's Vibration Needle.

The vibrations are observed first when the solid cylinders are in the middle of the case, and the hollow cylinders are at the ends, and again when the solid and hollow cylinders are interchanged in position. In this case, as is easily shown from (32) that, if m be the mass of each of the shorter hollow cylinders, m' the mass of each of the solid cylinders, and $2r$ the length of each,

$$\mu_1 - \mu_2 = 16\,(m - m')r^2.$$

The quantities on the right are all easily found, but the calculated result can hardly be relied on as very accurately the value of $\mu_1 - \mu_2$, on account of possible want of uniformity in the cylinders.

The moment of inertia μ of a given vibrator may also be determined by first observing the period T, then altering the moment of inertia by a known amount ν, and observing the period T'. If k be the same in both cases, or may be neglected, we have

$$\mu = \frac{\nu T^2}{T'^2 - T^2} \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Maxwell's Vibration Needle.

This process is frequently employed when the vibrator is of such form or dimensions that its moment of inertia cannot be found by calculation. A known mass of convenient figure, for example a couple of spheres, is made capable of being distributed symmetrically about the axis in two different configurations for which the difference of moment of inertia can be calculated. If change of total mass is of no consequence, a suitable known mass can be added to the vibrator in a convenient position, and the change of moment of inertia is then the moment of inertia thus added.

The torsional rigidity of a thin wire can also be determined by suspending a magnet of known magnetic moment (see below, vol. ii.) by a measured length of the wire so that the magnet rests with its length in the magnet meridian, and then twisting the upper end of the wire through a measured angle θ, which may be observed by means of a mirror attached to the magnet.

If H be the horizontal intensity of the field, M the magnetic moment of the magnet, then the couple tending to bring the magnet is (by the definitions of M and H) $MH \sin \theta$. The couple opposing return to zero is $\tau\theta/l$. Hence

$$\tau = MHl \frac{\sin \theta}{\theta} \quad . \quad . \quad . \quad . \quad . \quad (34)$$

The details of this method must be deferred.

Imperfection of Torsional Elasticity.

The torsional rigidity of a wire is, however, apart from the difficulty of its exact determination, a somewhat inconvenient means of obtaining an equilibrating couple. The zero position is subject to change even for moderate amounts of twist, in consequence of the slow working

out of a remainder of twist after removal of the deforming couple. This remainder is greater the longer the wire has been kept twisted. From this cause glass fibres are very unsuitable.*

Effect of Change of Temperature.

The effect of change of temperature for iron, copper, and brass has been accurately found by Kohlrausch.† Roughly speaking, the rigidity is diminished by $\frac{1}{2}$ per cent. for 10° rise of temperature. Change of temperature also changes the length of the wire, and thus alters the twist for a given angular deflection. This alteration is, however, only about $\frac{1}{10}$ of the change of rigidity.

Elastic Fatigue.

The torsional rigidity of a metallic wire is slightly diminished and its internal viscosity, and therefore the rate of subsidence of oscillation, greatly increased by what Sir W. Thomson ‡ has called *Elastic Fatigue* produced by continuous or frequent torsional oscillation.

Silk Fibre Suspensions.

In most cases it is preferable when possible to produce the equilibrating couple by some means independent of torsion. The mode of suspension generally adopted is by unspun clean silk-fibre, which combines great strength with very slight torsional rigidity. A single fibre (half of an ordinary cocoon fibre) will bear a weight of three or four grammes, and with a large margin of safety from half a gramme to one gramme. The torsional rigidity of fibres of Japanese silk of diameters from ·0009 to ·0015 centimetre varies (when expressed in terms of either force of a couple

* Mr. C. V. Boys (*Phil. Mag.* June, 1887) recommends fibres of quartz.

† *Pogg. Ann.* Ser. 3. Bd. CXLI. (1870), p. 481. See also *Encycl. Brit.* Art. "Elasticity," by Sir W. Thomson, § 79.

‡ "Elasticity and Viscosity of Metals," *Proc. R.S.*, May 18, 1865 or *Encycl. Brit.* Art. "Elasticity," § 30.

acting on an arm of one centimetre) from ·00091 to ·00250 dyne.* A silver wire of the same torsional rigidity would have a radius of from ·00022 to ·00028 centimetre, and would not bear more than about ·4 gramme without breaking.

The Bifilar Suspension.

The Bifilar suspension consists of two wires or threads, attached at their upper ends to two fixed points, and at their lower ends to two points in the suspended body. When the system is in equilibrium under the action of no applied forces, except such as are vertical, the threads are in a vertical plane and the centre of gravity of the body is in the lowest possible position. But if a system of couples acts on the body so as to turn it round a vertical axis, the threads no longer lie in a vertical plane, and a couple comes into play opposing the applied couple.

The suspension has the advantage of giving a perfectly definite zero of directive force, practically unaffected by the state of the fibres as to torsion, and but little changed by variation of temperature; and the couple given by it for a given deflection is capable of sufficiently accurate evaluation for many purposes by measurement of the dimensions of the arrangement. It is, however, used in many electric and magnetic instruments simply to give a reacting couple, and the constant of the suspension is, if necessary, determined by experiment.

General Theory of Bifilar.

We shall consider the perfectly general case of two threads of given lengths hung from given fixed points not necessarily in a horizontal line, and attached at their lower ends to a body suspended from them at two points

* T. Gray, "On Silk *v.* Wire Suspensions," *Phil. Mag.*, Jan. 1887.

General Theory of Bifilar.

at a given distance apart and in given positions relatively to the centre of inertia of the body. If the deflecting system of forces be a couple in a horizontal plane, the reaction of the bifilar must also be a couple in a horizontal plane, the projection in fact of the couple given by the horizontal components of the tensions of the threads, which plainly, since there is no horizontal resultant force to be equilibrated, must be equal and opposite. Let AA', BB', Fig. 42, be the respective projections on a horizontal plane of the upper and lower points of attachment of the fibres in the deflected position of the bifilar, and let $2a$, $2b$ be the lengths of

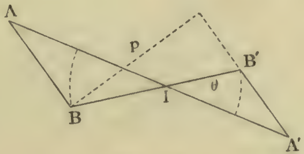

FIG. 42.

these projections. The horizontal forces act in the directions BA, $B'A'$, which are therefore parallel. If L be the moment of the couple due to the bifilar, and p the perpendicular distance between BA, $B'A'$, each horizontal component of tension is L/p.

If M be the whole mass suspended, the total downward force is Mg; and this must be equal to the sum of the vertical components of the two tensions. Hence if we denote one of these components by $\frac{1}{2}Mg(1 + c)$, the other must be $\frac{1}{2}Mg(1 - c)$. Again, if we denote the mean of the vertical heights of A above B and A' above B' by h, and call one of these $h(1 - e)$, the other

General Theory of Bifilar.

must be $h(1+e)$. We may suppose the tension $\frac{1}{2}Mg(1+c)$ to correspond to the vertical height $h(1+e)$, then if c be taken positive, e will be a positive or a negative quantity according as the greater tension corresponds to the greater or to the less vertical height. Hence, by the triangle of forces,

$$BA = h(1+e)\frac{L/p}{\frac{1}{2}Mg(1+c)}, \quad B'A' = h(1-e)\frac{L/p}{\frac{1}{2}Mg(1-c)}. \tag{35}$$

Adding, putting for $(BA + B'A')p$ its value $4ab\sin\theta$, where θ is the angle of deflection, and solving for L, we get

$$L = \frac{Mgab(1-c^2)}{h(1-ec)}\sin\theta \quad . \quad . \quad . \quad (36)$$

This equation shows that for a given deflection θ, L is smaller the smaller a and b, and the greater h, that is the sensibility is greater the closer the fibres and the greater their lengths.

Case of Symmetrical Arrangement.

In the actual use of the arrangement the lengths of the fibres are the same, the fixed points are in a horizontal plane, and the points of attachment to the suspended body are symmetrical with respect to the centre of inertia. Hence the pull is the same in both fibres, and e and c in (36) are both zero. Accordingly (36) reduces to

$$L = \frac{Mgab}{h}\sin\theta. \quad . \quad . \quad . \quad . \quad (37)$$

The value of h in this case is easily found. We have now from the figure $IB = IB'$, and $AB = A'B'$. But if l be the length of the fibre,

$$AB^2 = a^2 + b^2 - 2ab\cos\theta = l^2 - h^2;$$

and therefore

Case of Symmetrical Arrangement.

$$h^2 = l^2 - (a-b)^2 - 4ab\sin^2\tfrac{1}{2}\theta.$$

Hence from (37)

$$L = \frac{Mgab\sin\theta}{\{l^2 - (a-b)^2 - 4ab\sin^2\tfrac{1}{2}\theta\}^{\frac{1}{2}}} \quad . \quad . \quad (38)$$

If a be nearly equal to b, $(a-b)^2$ may be neglected; and if l be very great in comparison with either, both this term and $4ab\sin^2\frac{1}{2}\theta$ may be neglected. Under the latter condition

$$L = \frac{Mgab}{l}\sin\theta \quad . \quad . \quad . \quad . \quad . \quad (39)$$

Of course for small deflections θ may be put for $\sin\theta$ in all the formulas found for L.

Adjustment of Bifilar.

The accuracy of the adjustment for the fulfilment of the conditions $c = 0$, $e = 0$, may be tested and completed as follows: One of the fibres is raised a little relatively to the other by inclining the instrument. Supposing the adjustment to have been perfectly made, the effect of this would be, without sensibly altering e from zero, to make c appreciable; for one fibre being made more nearly vertical than the other, and having a horizontal component of tension equal to that of the other, must have a greater vertical tension. Hence the sensibility as measured by the ratio of a deflection θ to the opposing couple L will have been increased. Supposing the adjustment only *nearly* made, e will be very small, and any increase in the value of c will still increase the sensibility. On the other hand, if the effect of raising the fibre is to make the weights borne by the

Adjustment of Bifilar. two fibres more nearly equal, the result will be a diminution of the sensibility. Since the sensibility is proportional to the square of the period of vibration, any change in it will be at once shown if a small deflection is produced and the period roughly observed. If lifting one fibre a small distance diminishes the sensibility, while lifting the other increases it, more weight is borne by the latter than by the former, and the want of adjustment is to be remedied by shortening the fibre which bears the smaller weight, and lengthening the other until raising either fibre as a whole increases the sensibility.

Effect of Torsion. The total directive couple given by the bifilar is due not only to the raising of the suspended weight in consequence of the deflection, but is partly due to torsion. The correction is made simply by adding to the value of L already found the torsional couple for the two wires. Thus we have for the case of (39) if τ be the torsional rigidity of each wire,

$$L = \frac{Mgab}{l} \sin\theta + 2\tau\,\frac{\theta}{l}\,. \quad . \quad . \quad . \quad (40)$$

Effect of Flexural Rigidity. So far no account has been taken of any want of perfect flexibility of the threads, and when these are fibres of silk, no correction is really necessary. If, however, they have sensible flexural rigidity and the extremities be so fixed that they remain vertical in all positions of the suspension, each wire is bent into an elastic curve, of the shape roughly shown in the figure. The amount by which the deflection is diminished by the flexural rigidity is approximately the sum of the two small

distances $A'A$ and BB' in the figure; and it is easy to show that this is, very nearly, $2d\sqrt{B}/l\sqrt{T}$, where d is the whole horizontal displacement of the lower end of the fibre, T the vertical tension, and B the flexural rigidity of the fibre for bending in its plane.* Hence the couple derived from the observed deflection must be increased in the ratio of 1 to $1 - 2\sqrt{B}/l\sqrt{T}$; that is, the result is the same as if the length of the wire, whatever its amount, were diminished by $2\sqrt{B/T}$. Effect of Flexural Rigidity.

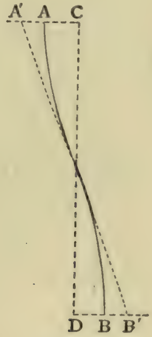

FIG. 43.

For consider one thread of the bifilar. Let the origin be at the centre, x be measured vertically downwards and y horizontally, and the tension at the centre be resolved into a vertical component T and a horizontal P; and let the inclination of the thread to the vertical be small. Equating the elastic couple at the section of coördinates x, y, which is $B \times$ curvature of thread

* That is, the product of the Young's modulus of the material into the moment of inertia of a cross-section (of unit mass per unit of area) round an axis through the centre of inertia at right angles to the plane in which the bending takes places.

Effect of Flexural Rigidity in Bifilar.

$= -Bd^2y/dx^2$, to the sum of the moments of T and P about that cross-section, we get the differential equation

$$B\frac{d^2y}{dx^2} = Ty - Px. \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

If a be put for $\sqrt{T/B}$ the solution of this equation is

$$y = \frac{P}{T}x + C\epsilon^{ax} + C'\epsilon^{-ax} \quad . \quad . \quad . \quad . \quad . \quad (42)$$

Now, approximately $x = \frac{1}{2}l$ at the lower extremity, and $= -\frac{1}{2}l$ at the upper, and at either $dy/dx = 0$. Hence differentiating in (42) and substituting we get

$$C' = -C = \frac{P}{Ta}\frac{1}{\epsilon^{al/2} + \epsilon^{-al/2}},$$

and therefore

$$y = \frac{P}{T}\left(x - \frac{1}{a}\frac{\epsilon^{ax} - \epsilon^{-ax}}{\epsilon^{al/2} + \epsilon^{-al/2}}\right) \quad . \quad . \quad . \quad . \quad (43)$$

For $x = l/2$ this is

$$y = \frac{P}{T}\left(\frac{l}{2} - \frac{1}{a}\frac{\epsilon^{2al/2} - 1}{\epsilon^{2al/2} + 1}\right).$$

If the fibre be long we may take as an approximation,

$$y = \frac{P}{T}\left(\frac{l}{2} - \sqrt{\frac{B}{T}}\right).$$

Hence we have finally

$$d = 2y = \frac{P}{T}\left(l - 2\sqrt{\frac{B}{T}}\right),$$

the result stated above.*

The amount of this correction is not generally negligible.† Its amount for a wire of $\frac{1}{200}$ cm. in

* See also Kohlrausch, *Wied. Annalen*, Bd. xvii. (1882).

† It may be calculated for a wire r centimetres in radius, and carrying a load of W grammes, by the formula $r^2\sqrt{2\pi E}/\sqrt{W}$, where E is the Young's modulus of the material in grammes weight per sq. cm. The value of E for copper is 1200×10^6, for silver 736×10^6, for iron 2000×10^6, for gold 813×10^6, for platinum 1700×10^6.

radius stretched by a tension of 100 grammes weight is ·22 cm. for copper, ·17 cm. for silver, ·18 cm. for gold, and ·26 cm. for platinum. [If only one end of the wire be constrained to remain vertical and the other end be straight the correction is of course only half of that just found.]

Comparison of Unifilar and Bifilar.

To compare the sensibilities of bifilar and unifilar suspensions of the same length and made of wire of the smallest possible diameter for the weight Mg to be carried, let S be the practical tenacity of the material—that is, the greatest weight per unit area which the wire will bear without experiencing inconveniently great permanent elongation. The least radius which can be used for the bifilar is given by the equation $r^2 = Mg/2S\pi$. For the unifilar the least radius possible is given by $r'^2 = Mg/S\pi = 2r^2$, and the torsional rigidity is $\frac{1}{2}\pi n r'^4 = 2\pi n r^4$. Hence by (40) for equal sensibility of unifilar and bifilar we have

$$2\pi n r^4 = Mga^2 + \pi n r^4.$$

Hence we get, putting for Mg, $2S\pi r^2$,

$$\frac{a}{r} = \sqrt{\frac{n}{2S}} \quad . \quad . \quad . \quad . \quad . \quad (44)$$

Taking the value of S as only $\frac{1}{8}$ of the utmost tenacity or breaking weight W, we get $a/r = 2\sqrt{n/W}$. The value of $\sqrt{n/W}$ is about 10 for silver, gold, and copper, and about 12 for platinum. Hence the bifilar is equal in sensibility to the unifilar (under the conditions stated as to diameter of wire), when the ratio of the distance

of the wires apart to the diameter of the wire is about 5 in the case of the first three metals, and about 6 in the case of platinum.

Effect of Change of Temperature.

The torsional term in the bifilar is the less important, and as the effect of change of temperature on the other term is due to expansion of the wires, the correction is very slight. If it were worth while, the balance might be compensated for effects of change of temperature by attaching the wires to a bar, the expansion or contraction of which would, by altering the distance of the wires apart, just annul the effect of change of length.

Determination of Constant of Bifilar.

The directive couple per unit of deflection given by the bifilar may be determined by the oscillation of a body of known moment of inertia as described above if the deflections are made small. Its determination from the dimensions of the apparatus cannot be done with accuracy unless the parts are made very large, owing to the difficulty of measuring the exact distance of the fibres apart. This has, however, been done by Kohlrausch in a very large bifilar balance made by him and used in an important series of determinations of the earth's horizontal magnetic force.*

Direct Determination of Couples.

Couples may also be directly determined with sufficient accuracy for many purposes by means of the following arrangement or some modification of it:—The suspended body is kept in equilibrium in the proper position by means of a bifilar suspension, or a single thread or thin wire under torsion. When a deflecting

* *Loc. cit.* and below (vol. ii.) in the chapter "On the Determination of *H.*"

Direct Determination of Couples.

couple acts on the body it is turned round a vertical axis, and is brought back to the initial position of equilibrium by means of two pendulums, the points of suspension of which are on sliding pieces which can be moved along horizontal parallel bars fixed above at right angles to the plane of the bifilar when in the equilibrium position, or to some fixed plane through the unifilar. Each pendulum cord has attached to it a thread which pulls symmetrically on the two sides of the suspension. When the body is deflected, the sliding pieces are moved in opposite directions, so that, in consequence of the opposite inclinations of the pendulums to the vertical, forces restoring equilibrium are applied to the body. Let the deflecting couple be L. Supposing the two points of suspension of the pendulums to be on one level, and the points of attachment of the pulling threads to the pendulum cords to be on a level lower by a distance of l cm., and at a distance of d cm. apart, the distances of the verticals through the points of suspension from the corresponding verticals through the attachments of the threads to the pendulum cords to be d_1, d_2 cm. for the respective pendulums, and the mass of each bob W grammes, we have, for the moment of the equilibrating couple, the value

$$Wg\,\frac{d_1 + d_2}{l}d.$$

Hence, equating moments, we get

$$L = Wg\,\frac{d_1 + d_2}{l}d. \quad . \quad . \quad . \quad . \quad (45)$$

CHAPTER V.

ELECTROMETERS.

Definition of Electrometer.

An electrometer has been defined as an instrument for measuring differences of electric potential by means of the effects of electrostatic force. It consists essentially of two conductors, between which is established the difference of potentials which it is desired to measure. The electrostatic force set up produces motion of the parts of one of these conductors relatively to one another, or motion of the conductor as a whole relatively to the other conductor; and from this motion, or from the mechanical force which must be applied to restore and maintain equilibrium in the configuration of zero electrification, the difference of potentials is inferred. We shall call this conductor the *Indicator* of the instrument.

Absolute Electrometer.

When the instrument contains within itself a means of comparing the electric force called into play with other forces known in amount, as for example the force of gravity on a given mass, or the elastic reaction of a stretched spring, it gives directly by its indications the value in absolute electrostatic units of the difference of potentials measured, and is called an *Absolute Electrometer*.

When the instrument gives only comparisons of the

electrostatic forces with other forces the amount of which it does not itself contain any means of determining, its indications can only be obtained in absolute units by a comparison with those of an absolute electrometer.

Non-Absolute Electrometer.

When the mode of variation of these undetermined forces is known and remains constant, one such accurate comparison is sufficient to allow a coefficient to be determined by which results proportional to measured differences of potential must be multiplied for reduction to absolute measure. This coefficient is called the *Constant* of the instrument.

Constant of Electrometer.

Electrometers have been divided by Sir William Thomson into three classes:—

Classes of Electrometers.

I. Repulsion Electrometers.
II. Attracted Disc Electrometers.
III. Symmetrical Electrometers.

To the first class belong instruments in which a difference of potential between the indicating conductor and the fixed conductor is shown by relative motion of parts of the indicator produced by their mutual electric repulsion. In such an instrument, when properly constructed, the fixed conductor either is a closed or nearly closed conductor, or is in conducting contact with and contained within a closed conductor, which also contains the indicator.

Most of the instruments known as electroscopes, for example those of Cavallo, Volta, Bennet, and Henley, act thus by electric repulsion; but as these cannot, however made, be regarded as accurate-measuring

instruments, we do not here give any description of them.

Coulomb's Torsion-Balance.

The first accurate repulsion electrometer devised was Coulomb's Torsion-Balance, which gave good results in the hands of Coulomb himself and of Faraday; but this instrument has been almost entirely superseded by much more delicate and convenient electrometers chiefly belonging to the other two classes. As, however, it is capable of being made to give fairly accurate results in absolute measure, we give here a short account of its construction and use. It is essentially a modification of the torsion-balance previously employed by Michell and Cavendish in experiments on the gravitation of matter.

The instrument is represented in Fig. 44, which also shows some of the principal pieces, detached and on a larger scale. The indicator is a thin rod of shellac, carrying at one end a small pith-ball, b, as smoothly gilded as possible, and at the other end a paper disc, d. This rod is suspended horizontally with the paper disc vertical within the cylindrical case $BACD$, as the movable arm of a torsion-balance, by a fine silver wire attached to a button, K, which is supported on and turns round the graduated torsion head MM', carried by the upright tube shown standing on the case in the figure. A cylindrical piece, attached to and coaxial with MM', fits into the cylindrical piece H, which is cemented into the upper end of the glass tube. The silver wire is attached to the button K above and to the indicator below by two small porte-crayon clamps, and is as nearly as may be in the axis of the case. A

continuation of the lower clamp downwards below the horizontal rod, by giving stability to the indicator, secures the verticality of the plane of the paper disc, which thus serves as an air vane to damp the motion of the

Coulomb's Torsion-Balance.

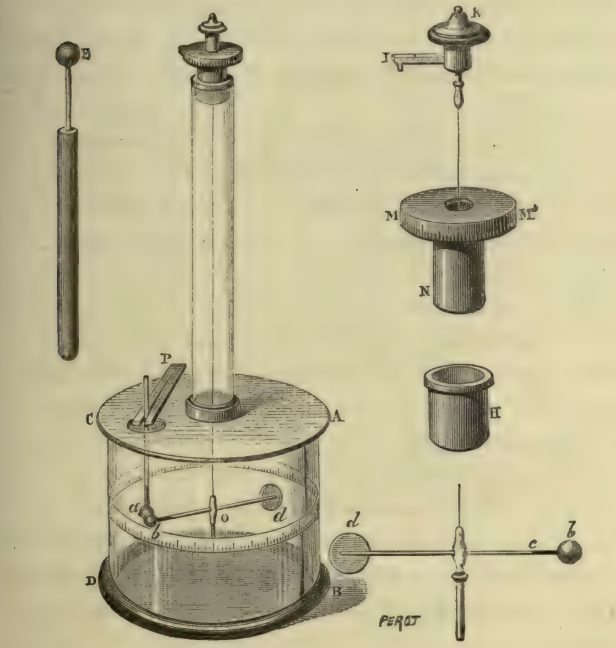

FIG. 44.

indicator. Round the case, opposite the indicator, is a scale graduated to degrees, by which the position of the indicator can be determined. In the cover of the case, at a distance from the centre equal to the distance of

Coulomb's Torsion-Balance.

the ball b from the suspension wire, is a hole through which a thin rod of sealing-wax, bearing another gilded ball, a, of pith, can be passed. This rod is attached to a holder, P, so that the ball a when in position is on a level with b and has its centre exactly on the radius through the zero division of the scale; and the holder is provided with a geometrical fitting to the cover, which ensures its being always replaced in exactly the same position.

In Coulomb's instrument the wire was about 70 cm. in length and ·035 mm. in diameter, and a couple equal to that due to the weight of 1·7 of a milligramme acting at the extremity of an arm of 1 centimetre sufficed to maintain the wire twisted through 360°.

Mode of using the Torsion-Balance.

When the instrument is to be used, a is withdrawn, and the button K turned until the ball b rests with its centre opposite the zero of the lower scale, and MM' is turned until the index is at zero. The wire now exerts no torsional couple. The ball a is then replaced, and b rests against it slightly deflected from its previous position. The instrument is now adjusted ready for use. As an illustration of how the instrument was used by Coulomb, we shall describe an experiment by which he determined the repulsion of two electric charges.

Experiment from which Coulomb deduced Law of Distances.

The adjustment of the ball b was first made, then the ball a was electrified by being brought into contact with the previously charged ball E, and placed in position. The charge was shared with b, which was therefore repelled, and the indicator took up a new position of equilibrium, making an angle of 36° with

Experiments from which Coulomb deduced the Law of Distances.

the former, as shown by the lower scale. The wire then had a torsion of 36°. Coulomb now turned the wire by means of the button K until he had diminished the deflection to 18°, and found that he had turned the button round through 126°, thus increasing the torsion of the wire to 144°, or four times its previous amount. He then diminished the deflection by the same means to 8°·5, and found that the total torsion was 575°·5 or almost sixteen times its initial amount.

Coulomb's Law of Torsion.

Coulomb had previously verified by independent experiments the important fact that the couple required to hold an elastic wire twisted through an angle not so great as to overstrain the wire varied directly as the angle; and he concluded that, as the distances between the centres of the balls were approximately in the ratio 4 : 2 : 1, the forces of repulsion in the respective cases were in the ratio of 1 : 4 : 16, that is, inversely as the square of the distance.

Theory of the Experiment.

If we could regard the electrification of the balls as the only electrification concerned, and the distribution in each case as uniform, the exact theory of the experiment would be as follows:—Let r be the radius of the circle in which the indicator ball moves, a the deflection of the indicator from zero, θ the total angle through which the upper end of the wire is turned relatively to the lower, l the length of the wire, and F the force of repulsion between the balls for the deflection a. The moment of F round the axis is $Fr\cos\frac{1}{2}a$. Assuming that F varies inversely as the square of the distance, and denoting the charges by q, q', we have, since $2r\sin\frac{1}{2}a$ is the distance between the balls,

Theory of the Experiment.

$F = qq'/4r^2 \sin^2 \frac{1}{2}a$, and the couple becomes $qq'/4r \sin \frac{1}{2}a \tan \frac{1}{2}a$.

If τ denote the torsional rigidity (see p. 231 above) of the wire, that is, the couple required to maintain a twist of one radian per unit of length, we have $\tau\theta/l$ for the torsional couple applied by the wire. Equating this to the former couple we get—

$$\theta \sin \tfrac{1}{2}a \tan \tfrac{1}{2}a = \frac{lqq'}{4r\tau}. \quad . \quad . \quad . \quad . \quad (1)$$

Divergence of Results from Law of Inverse Square.

Hence on the suppositions made the quantity on the left should, if the law of the inverse square is true, be constant for constant charges and different values of a and θ. The comparison gives the following table:—

a	θ	$\theta \sin \frac{1}{2}a \tan \frac{1}{2}a$
36	36°	3·614
18	144	3·568
8·5	575·5	3·169

The numbers in the third column fall off considerably with diminution of distance. There ought to be a falling off on account of induction between the two balls, and to this no doubt the discrepancy is in great measure due.

Comparison of Charges of Carrier Ball.

The instrument may also be used for the comparison of different charges of the ball a. All that is necessary is to maintain a constant charge in the ball b, and find the torsional couples which must be applied to produce a given constant deflection in the different cases. The charges of a may be made proportional (p. 100 above) to the varying density of the distribution at different points

on a conductor, and thus the law of variation of the density may be experimentally obtained. A large number of experiments have been made in this manner by Coulomb, Riess, and others.*

Effect of Electrification of the Case.

In the form in which it was used by Coulomb, the torsion balance is an essentially inaccurate measuring instrument. The electrification of the glass case must have frequently affected the results to a considerable extent, and could not have been allowed for, even if it had remained constant during one set of experiments, without a very difficult experimental determination of the electrical state of the glass. If the interior of the case had been covered with a conducting coating furnished for purposes of observation with openings partially closed with wire gauze, all effect of exterior electrification would have been avoided, and the only effect due to the case would have been that of a perfectly definite electrification depending only on the position and charges of the interior system.

Calculation for a Spherical Conducting Case.

For example, the effect of this electrification can be easily calculated to a first approximation for a spherical case with an internal lining of conducting material. We have only to find the effect on b of the image of a in the spherical surface. If the radius of the surface be r_1, and the distance of the centre of the ball a from the centre r, the image of the charge q (regarded as a point-charge at the centre of a) is in the radius through the zero of the scale at a distance r_1^2/r (p. 78) from the centre, and its charge is $-qr_1/r$. The square of the

* Coulomb's *Memoirs*, and Mascart's *Traité d'Électricité Statique*, Tome 1, chap. iv.

Calculation for a Spherical Conducting Case.

distance of this image from b, when the deflection of the indicator is α, is $r_1^4/r^2 + r^2 - 2r_1^2 \cos \alpha$. Hence the force between the image and b is $-qq'r_1r/(r_1^4 + r^4 - 2r_1^2r^2 \cos \alpha)$, and the moment of this round a vertical axis through the centre is $-qq'r_1^4r \sin \alpha/(r_1^4 + r^4 - 2r_1^2r^2 \cos \alpha)^{\frac{3}{2}}$. This, added to the formerly found moment, gives the total moment balanced by the torsional couple.

The torsional rigidity of the wire must be determined if the results are to be obtained in units unaffected by the unknown elasticity of the suspension. Full details of the method of finding this constant are given in the last chapter.

Thomson's Electrometers.

For all purposes of accurate measurement, electrometers of the other two classes are more convenient. Most of these have been invented by Sir William Thomson, and his instruments or modifications of them are now in almost universal use. Accordingly we shall not occupy space with the description of instruments such as Dellmann's and Peltier's electrometers, or even the repulsion instruments of Thomson, although these, in the hands of the inventors and others, have yielded valuable results, but shall at once proceed with the description of the forms now adopted.

Attracted-Disc Electrometer of Snow-Harris

In an attracted-disc electrometer the attraction between two parallel discs at different potentials, and at a given distance apart, is compared with the elastic reaction of a stretched spring or the weight of a given mass. The first instrument of this kind was constructed by Sir William Snow-Harris, and used by him in some important experiments. It is shown in Fig. 45. In it

Attracted Disc Electrometer of Snow-Harris.

the indicator is a plane disc d suspended as one scale of a balance above a similar disc a, connected with the interior coating of a Leyden jar J, which is to be tested. The other scale P of the balance is weighted so as to equilibrate d when there is no electrification. When a

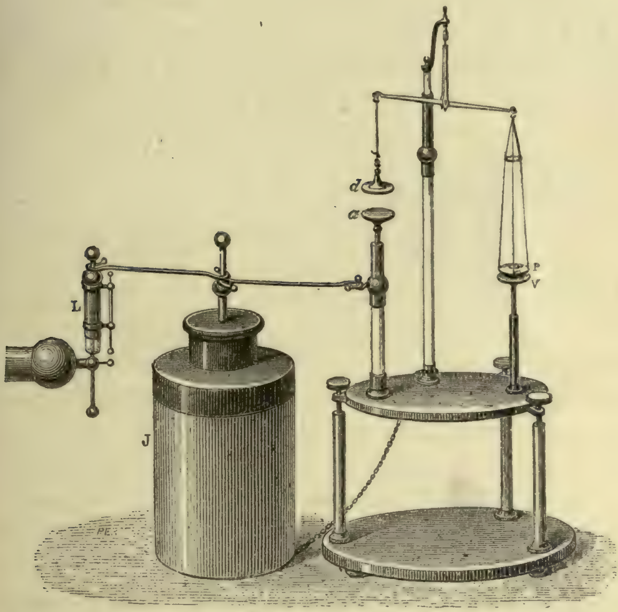

FIG. 45.

is charged, d is attracted, and equilibrium is restored by placing weights in P. The downward pull on d in the position of equilibrium is thus measured in absolute units from the known force of gravity on the mass

Attracted Disc Electrometer of Snow-Harris.

placed in P. The arrangement marked L is a unit jar arrangement which was used in the experiments of Snow-Harris to give a rough approximation in arbitrary units to the charge given to the jar.

This form of electrometer, though exceedingly defective in many respects, contains in a rudimentary form the principle of an attracted-disc electrometer.

Thomson's Attracted-Disc Electrometer, with Guard-ring.

Fig. 46 shows an improved arrangement in which the attracted disc is surrounded by a guard-ring, so

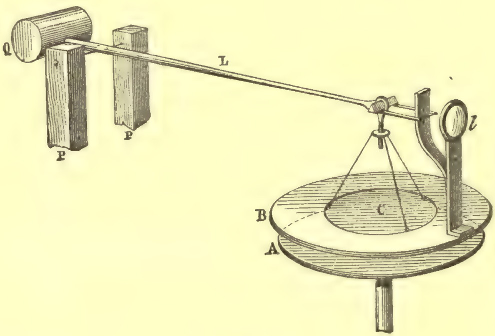

FIG. 46.

that the distribution of electricity over it may be regarded as approximately uniform. The indicating disc C is hung by metallic wires from one end of a metal beam pivoted on a horizontal wire stretched between the pillars PP, and twisted so that the torsion and the counterpoise Q tend to raise the disc C. It is necessary to apply a downward force to C to bring it into the plane of the guard-ring, and this force is de-

termined by placing weights on the disc until it has been depressed so as to bring the lever into what is called the "sighted" position. The lever is forked at the end to which the disc is attached, and the extremities of the prongs are joined by a horizontal black hair, which, when the lever turns, ascends or descends in front of a white enamelled metal plate carried by the stand of the lens l. On the enamel, one above the other, at a distance apart rather less than the thickness of the hair, are two well-defined black dots, and the hair is in the sighted position when it is symmetrically placed with respect to these dots. The hair and dots are viewed by the plano-convex lens l which is placed at a little less than its focal distance from the dots and hair. The hair plays, without touching, as close as possible to the enamelled plate, which is slightly convex in front, so that the hair is nearer to it when in the sighted position than when elsewhere. To obtain as much magnification as possible, the lens is placed at a distance from the hair only a little less than the principal focal distance, and the eye at a distance from the lens of 20 centimetres or more. To avoid parallax the lens is used with its convex side turned towards the hair, and the eye is moved up or down until the hair seems straight in the middle and to widen out at the ends equally above and below. If the eye is too high or too low, the lens will show the hair curved upwards or downwards.

Details of the Indicator

A very slight motion of the hair from the central position between the spots is possible with this arrangement. Sir William Thomson and Dr. Joule

Details of the Indicator.

have corrected so small a deviation as 1/50000 of an inch.

The disc nearly fills the aperture in the guard-ring, and it can be shown that its effective area, reckoned as uniformly charged on the side turned towards the disc B, is approximately the mean of the areas of the aperture and the plate A.* The disc and guard-ring are electrically connected by a wire which joins the guard-ring with the metal pillars on which the lever is mounted.

The attracting plate A is carried on an insulating pillar attached to a micrometer screw by which the plate can be moved upwards or downwards.

Method of using the Electrometer.

The method of using the electrometer is as follows. A constant difference of potentials is maintained between one of the plates, say the guard-ring and disc, and the earth, and the other plate A is connected to earth. The latter is then screwed up or down until the attracted disc comes into the sighted position, and the reading of the micrometer-screw is taken. The plate A is then connected to the body to be tested, and again moved up or down until the attracted disc comes into the sighted position.

If V be the difference of potentials between the earth and the guard-ring and disc B, and V' the difference of potentials between the plates when the movable plate A is connected to the body to be tested, d and d' the corresponding readings on the micrometer (so that $d' - d$ is the distance through which the plate has been

* See Maxwell's *El. and Mag.* vol. i. second edition, p. 307, for a closer approximation.

moved between the two readings), and F the force which must be applied to the disc to bring it to the sighted position, we have by (63), p. 58 above, Method of using the Electrometer.

$$V' - V = (d' - d)\sqrt{\frac{8\pi F}{S}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This is the difference of potentials between the body to be tested and the earth, and is obtained in absolute C.G.S. units of potential if $d' - d$ be taken in centimetres, S in square centimetres, and F in dynes.

This method of proceeding was adopted by Sir William Thomson, as it makes the result depend only on a determination of the difference of the distances of the plates apart in the two positions, and not on a determination of the actual distance of the plates apart, necessarily inaccurate on account of the difficulty of making the planes parallel and perfectly plane, which would have to be depended on if V were made zero and the position of A found for which the disc was depressed to the sighted position.

Sir William Thomson has called this mode of using the electrometer *heterostatic*, from the fact that an electrification independent of that to be tested is maintained on the plate B. When the electrification to be tested is alone made use of, as would be the case in the other mode of proceeding just stated, the instrument is said to be used *idiostatically*. "Heterostatic" and "Idiostatic" modes of using an Electrometer.

The final form of the absolute electrometer is shown in Fig. 47. The attracted disc and plates are contained within a cylindrical case of white glass, which is fixed by a brass mounting round its lower end to a horizontal Complete Form of Thomson's Absolute Electrometer.

Complete Form of Thomson's Absolute Electrometer.

sole-plate of iron, supported on three feet with levelling screws, and is closed above by a stout brass plate screwed to a brass ring cemented round the upper end.

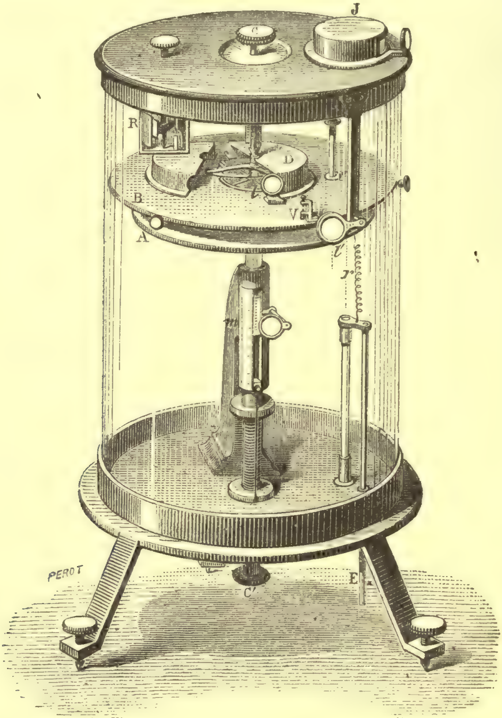

Fig. 47.

The height from sole plate to cover is 50 centimetres, and the diameter is 30 centimetres. The sides of the case, with the exception of apertures to permit obser-

vations of the interior points to be made, are coated inside and outside with tin-foil nearly as high as the plates which are in the upper part of the jar.

Complete Form of Thomson's Electrometer.

The case thus forms a Leyden jar, the coatings of which can be brought to any necessary difference of potentials. The guard-ring B is connected with the interior coating by its supports, which are metal pieces cemented to the inner side of the jar and covered with tin-foil. Within the jar, on the sole-plate, is placed a leaden tray containing pumice moistened with sulphuric acid, which maintains a dry atmosphere within the jar.

The Attracting Plate.

The attracting plate A is a stout plate of brass, with pieces cut out of it to allow it to pass the supports of the guard-ring, and is supported by an insulating stem of white glass cemented to a vertical pillar of brass, which is moved up and down in V guides by the micrometer screw C', working in a fixed nut in the sole-plate. The amount of vertical motion is observed by means of the vertical scale m, and a circular plate graduated on its edge, which projects from the screw and turns in front of a fine vertical wire. The former shows the number of complete turns made by the screw, the latter allows any fraction of a turn to be estimated to a degree of accuracy depending on the fineness of the graduation, and the precision with which the position of the wire on the circle can be read. The actual distance traversed is got from this result by multiplying by the step of the screw, which, in the first instrument made, was $\frac{1}{50}$ of an inch.

The attracted disc is made for lightness of thin

The Spring-Balance.

aluminium strengthened by a thick rim and radial ribs on its upper side, and is made as nearly as possible perfectly plane on its under side. Instead of being hung from one arm of a balance like the disc shown in Fig. 46, it is supported by three delicate springs, similar in shape to coach-springs, of which one only is shown in Fig. 47, projecting from underneath the cover *D*. These springs are placed symmetrically round the disc and meet at their points of crossing above and below The disc is attached to the lower point of crossing, and the upper point of crossing is attached to the lower end of an insulating stem carried at its upper end by a brass tube which slides in *V* guides, and can be moved up and down by the head *C* of a micrometer screw similar to that already described as moving the attracting plate *A*. Underneath this screw-head and fast to it is a micrometer circle, which serves to determine fractions of a turn, while complete turns are shown by the divisions on a vertical scale. The actual distance through which the disc is moved in any given case need not be known; all the upper micrometer screw gives is merely a comparison of distances.

Two small uprights stand on the centre of the disc, and between these is stretched a fine black hair, of which an image is formed in the conjugate focus by the achromatic lens *l*. The lens is so adjusted that this focus is between two screw points *V*, which are so placed as to touch the image above and below when the disc is in the sighted position. This image is observed through an eye-lens *l′* attached outside the jar to the brass mounting, and then, since the points and

the image of the hair are in focus in the same position of this lens, there is no error due to parallax. The Spring-Balance.

The attracted disc and springs are inclosed within the metallic box *D* (of which one-half is shown displaced) to prevent disturbance by external electrification. The hair is seen through a hole cut in the box opposite the lens.

Gauge for testing Hetero-static Electrification.

The difference of potentials between the inner and outer coatings of the jar is tested by an auxiliary attracted disc electrometer used idiostatically. This electrometer, which is called the *gauge*, is contained

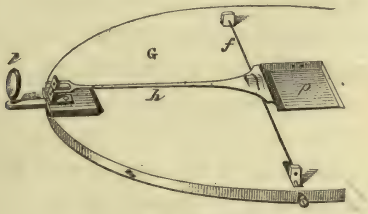

FIG. 48.

within the cylindrical box *J* on the cover of the jar. The arrangement is shown in detail in Fig. 48. The disc is a square piece of aluminium forming a continuation of a lever *h* of the same metal. This lever is forked and the prongs joined by a black opaque hair which moves in front of an enamelled plate on which are two black dots as already described. The position of the hair is seen through the plano-convex lens *l*, which is carried by a sliding platform attached to the guard-ring *G*. Instead of the counterpoise shown in

Gauge for testing Hetero-static Electrification.

Fig. 46, torsion given to the platinum wire *f*, to which the lever is attached in the manner shown in Fig. 48, and round which the lever turns as a fulcrum, forces the disc upwards. This upward force is balanced when the hair is in the sighted position by electric attraction between the disc and a parallel plate below it, which is in contact with the interior coating of the jar while the guard-ring and disc are in contact with the exterior coating. The attracting plate below is mounted on a fine screw, by which its distance from the disc and therefore the sensibility of the gauge can be varied at pleasure within certain limits. The sensibility of the gauge varies with any alteration in the elasticity of the torsional spring *f*. This however is of little consequence as the variations are not sudden, and it is never necessary to know the actual potential of the jar.

Between each end of the wire *f* and the supporting block is interposed a *U* shaped spring (not shown in Fig. 48) made of light brass. The end of the wire is attached to the extremity of one limb of the *U*, a pin passing through the supporting block to the extremity of the other limb. The two pins, the extremities of the springs, and the wire are in line. The springs can be turned round the pins as axes, so as to give any initial torsional couple to the wire which may be required, and by their spring cause the wire to be stretched with a nearly constant force.

The mode of attachment of the wire to the lever *h*, deserves notice. The wire is threaded through two holes in the broader part of the lever, near the square

disc, so that the part between the holes is above the lever. Halfway between the holes it passes over a small ridge piece of aluminium, which prevents the lever from turning round without twisting the wire.

Gauge for testing Heterostatic Electrification.

The plate A when the instrument is used is connected with the body to be tested by the electrode E, which passes through a plug of clean paraffin fixed in an aperture in the sole plate. The wire r completes the connection between E and A.

The Replenisher.

The difference of potentials between the coatings is kept nearly constant by means of a small induction machine R, called by Sir William Thomson the *Replenisher*. The construction and action of this part will be easily understood from Figs. 49 and 50; Fig. 49 shows the mechanism full-size in perspective elevation; Fig. 50, the same in plan.

Its Construction.

Two similar metal carriers C, D, each part of a cylindrical surface, are fixed on a cross-bar of paraffined ebonite so as to be slightly noncoaxial but inclined at the same angle to the cross-bar. Through the cross-bar and rigidly fixed to it, passes a cylinder of ebonite having at its ends metal pieces which form the extremities of an axle. The carriers turn round this axle within the circular cylinder marked out by the cylindrical metallic pieces A, B, which are insulated from one another and act as inductors. A receiving spring s or s', projects obliquely inwards through a hole in each inductor, with which it is also connected at the back, and is bent so that the carriers touch the springs on their convex sides, and pass on but little impeded by the friction. Two contact springs S, S', connected by a metallic arc project slightly

Its Construction.

inwards beyond the inductors so that the carriers, while opposite the inductors, come into contact with these two springs at the same time, and are therefore put into conducting contact. One of the inductors, *A*, is connected

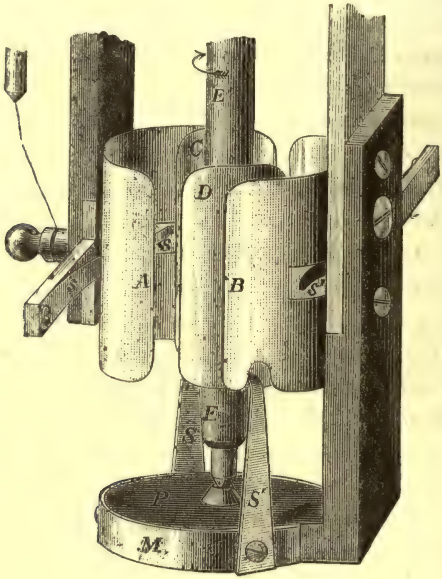

FIG. 49.

to the inner coating of the jar, the other, *B*, is attached by the supporting plate of brass to the cover of the instrument and therefore to the outer coating. A milled head attached to *E* projects above the cover and forms a handle by which the carriers are twirled round.

The electrical action is easily understood. An initial charge has been given to the jar, so that a difference of potentials exists between the coatings, the interior for example being positive. When the carriers come into contact with the springs S, S', they are brought to the same potential, and, since they are under the influence

Its Electrical Action.

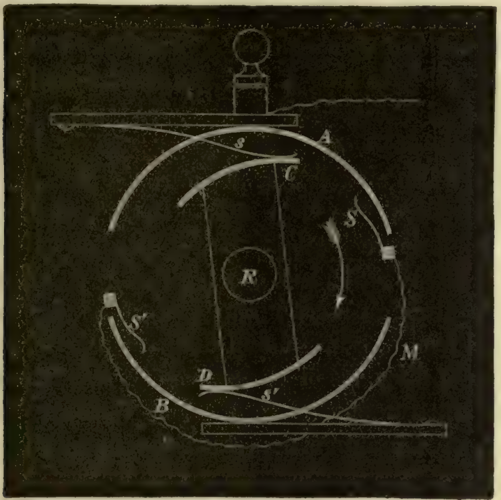

FIG. 50.

of the inductors, one carrier becomes charged positively, the other negatively. Then, turning in the direction of the arrow, they come into contact with the receiving springs, and being each (electrically) well under cover of the corresponding inductor, they give up the greater part of their charges, thus increasing the difference of potentials between the inductors.

If the carriers are turned in the opposite direction the action is of course reversed, and the difference of potentials is diminished. When the replenisher is not in action the carriers are not in contact with any of the springs.

Method of using the Absolute Electrometer.

The method of using the Absolute Electrometer is practically the same as that described for the more rudimentary instrument of Fig. 46. The force required to depress the disc against the action of the springs without overstraining is determined by removing the top cover of the jar and the cover of the balance, and loading the disc as symmetrically as possible with weights, while all electrical force is avoided by putting the electrode of the plate A in contact with the guard-plate B. The micrometer-screw C is then turned until the disc comes again into the sighted position, and the distance through which the plate was depressed is obtained from the initial and final micrometer readings in terms of divisions of the scale. (It was found in the original instrument made for Sir William Thomson that $\frac{6}{10}$ of a gramme depressed the disc through two divisions of the vertical scale and a fraction of one division on the graduated disc.) Several determinations of this distance are made at different temperatures to obtain data for the elimination of the effects of temperature on the springs. The weights are now removed, the covers replaced, and the instrument is ready for use in absolute measurements.

When it is to be thus used the guard-ring and attracting plate are put into conducting contact by connecting the electrode of the latter with the charging

rod let down through the aperture provided for it in the cover, and the disc is put accurately into the sighted position. It is then raised by the micrometer screw through a distance for which the force F has been determined. To bring it back to the sighted position will require the application of that force. The jar is next charged to the degree determined by the sensitiveness of the gauge, and the potential kept constant by using the replenisher as described. The attracting plate is now connected by means of its electrode with the exterior coating of the jar, and the micrometer moved up or down until the disc is brought into the sighted position, when the micrometer reading is taken. This is called the *earth-reading*. The electrode of the attracting plate is now brought into contact with the conductor whose potential is to be tested, and the plate again moved by the micrometer until the disc is once more in the sighted position and the reading once more taken. The difference between the two readings gives $d'-d$ of (2), p. 265 above, which since F has been determined, and S is supposed known, gives in absolute units the difference of potentials $V'-V$ between the conductor tested and the outer coating of the electrometer jar.

Method of using the Absolute Electrometer.

Sir William Thomson has also constructed an attracted disc electrometer capable of being easily carried about from place to place, and therefore adapted for observations of atmospheric electricity at different places in rapid succession by the same observer, for use by explorers, or for any purpose for which smallness of size and portability are necessary.

Portable Electrometer.

T 2

Arrangement of Parts.

The arrangement is precisely similar to that of the gauge, Fig. 48, turned upside down. The guard-ring is below the attracting disc, and the trap-door and lever of aluminium, with the connected sighting apparatus, suspended as described above, are on the under side of the guard-plate, and are therefore protected from electrical action from above. The lever and fork are

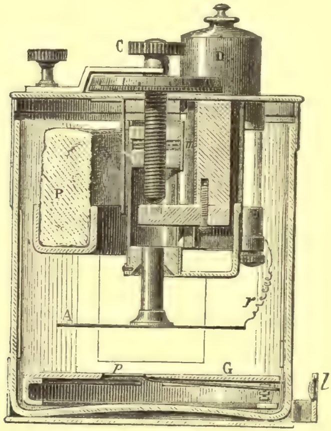

Fig. 51.

further protected from electrical action by a cage of thin wire surrounding them. Since the instrument is not set up and levelled like the absolute electrometer, the piece, consisting of the trap-door and lever, is carefully balanced about its platinum wire axis, so as to be unaffected by gravity, and is therefore brought to the

sighted position by the same electric force whatever the position of the instrument.

Arrangement of Parts.

The plates with the lever, &c., are contained, as in the absolute electrometer, within a glass jar coated inside (except where apertures are left to allow the interior to be seen) with tinfoil. A protecting brass case, in which are openings corresponding to those in the interior coating, forms the exterior coating. The jar is charged as in the absolute electrometer, and its interior coating is in electrical contact with the guard-ring, which is therefore at the same potential. The instrument is thus used heterostatically. The dimensions of the case are, diameter 9 centimetres; height, 10·5 centimetres.

Micrometer and Geometrical Slide.

The attracting plate is mounted on an insulating rod of glass, attached to a brass tube which can be moved along *V* guides by a micrometer screw, of step $\frac{1}{50}$ of an inch, working in a nut fixed in the upper end of this tube. The details of this arrangement are shown in the figure. A strong frame projects downwards from the cover of the jar and bears the *V* guides in which the brass tube moves. A spring pressing against the tube opposite each *V*, or a single spring between the *V*s, causes the tube to bear on the guides. The lower end of the micrometer screw is shod with a convex piece of hard polished steel, which bears upon an agate-faced platform rigidly attached to the framework, and projecting into the tube through a slot of sufficient length to give the necessary range of motion.

To prevent lost time and side-shake, a guide-nut fitting well but easily in the brass tube is placed above

Micrometer and Geometrical Slide.

the effective nut, and is prevented from turning round by a piece which projects from it through a slot in the tube. A spiral spring between the nuts presses them apart and thus keeps the upper side of the screw thread in contact with the nut. A fork, projecting from the screw shown on the left side of the cover in the diagram over the micrometer circle, prevents the screw from being pulled up through more than a distance of about $\frac{1}{20}$ of an inch.

The micrometer screw is turned by a head above the case, and angles turned through are read on a vertical scale and a graduated circle turning relatively to a fixed mark on the cover. The vertical scale is engraved on two fixed cheeks parallel to the axis of the screw, which are in plane with an index piece which slides up and down freely between them. Increasing numbers on the scale correspond to increasing distance between the plates, and the zero of the scale nominally corresponds to zero distance, though no particular care is taken to make it actually so correspond. The graduated circle is divided into 100 equal parts so that each corresponds to an angle of $3^{\circ}{\cdot}6$ turned through by the screw; and the distance between two consecutive divisions on the vertical scale is equal to the step of the screw, so that the index advances one division in each complete turn of the screw.

The hole in the roof of the instrument through which the screw passes is made large enough to allow the screw to pass without touching, and the graduated circle which covers this hole above is raised a little above the cover so as not to touch it.

The attracting plate A is connected by a light wire r with a brass plate supported by a glass column from the roof. This brass piece is continued upwards by the main electrode of the instrument, a stout wire passing without contact through an opening in the cover, and carrying the cap D, which can be moved up or down along it through a short distance. This, when in its lowest position, connects the electrode with the outside of the case of the instrument and closes the opening through which the electrode passes, and when raised serves as an umbrella to protect the electrode from wind and rain.

Attracting Plate and Electrode.

A lead tray attached to the roof supports a block of pumice moistened with sulphuric acid which preserves a dry atmosphere within the case. A caution, "Pumice dangerous if not changed once a month," is engraved on the cover beside a small holder for a card, on which the dates of the renewal of the pumice are to be noted.

Mode of using the Portable Electrometer.

The mode of using the instrument is heterostatic and similar to that described above for the absolute electrometer. The jar is charged by means of an electrode let down through a hole in the cover, and a negative charge is given, so that increased readings of the micrometer correspond to increased positive charges on the attracting plate. When an experiment is to be made the umbrella is put down, the disc brought to the sighted position by the micrometer, and the reading (which we call the first earth-reading) taken. The umbrella is then raised and the body to be tested connected to the electrode. The disc is again brought

Mode of using the Portable Electrometer.

into the sighted position and a reading, D' say, obtained. The body is then disconnected and a second earth-reading taken. The times at which the readings are taken are noted. In consequence of leakage of the charge of the jar the second earth-reading may differ from the first, and the earth-reading for the time at which D' was taken is to be estimated from the two earth readings. Let this reading be denoted by D, and the difference of potentials between the body and the case by V. We have—

$$V = m(D' - D) \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where m is the average value, in absolute units of potential, of a scale division for the range between D and D'. This value depends on the elasticity of the spring suspension of the trap-door and lever, the area of the trap-door, and the scale of graduation adopted; and does not depend on the potential of the jar or on the electrification tested, except in so far as the smallness of the attracting plate causes the electric field between it and the trap-door to deviate sensibly from uniformity at the greater distances. The constant m can be determined of course by an experiment with a known difference of potentials, and this ought to be done for different parts of the scale. The range of the instrument is 15 turns of the screw, or about 5,000 volts potential; that is approximately the difference of potentials between the poles of a battery of 5,000 Daniell's cells arranged in series.

The portable electrometer has certain faults, which are, however, mostly due to its smallness of size. The

capacity of the jar is not large enough to prevent the potential of the jar from being sensibly affected by the electrification of the attracting plate; the suspension is affected by change of temperature; the wire guards surrounding the aluminium lever do not sufficiently protect it from electric influence; and, as already stated, the plates are not large enough to ensure that the value of a division of the scale may be the same in every part of the scale.

Sources of Error in Use of Portable Electrometer.

These sources of error, except that due to temperature, have been corrected in a large instrument on the same principle, and on the whole similar in construction, which Sir William Thomson has made and called the Standard Electrometer. For a detailed description of this instrument the reader is referred to Sir W. Thomson's Reprint of Papers on Electrostatics and Magnetism.

Long Range Electrometer.

Sir William Thomson has modified an arrangement of the portable electrometer and enlarged its size so that a reliable instrument with a range from about 4,000 volts to 80,000 volts is obtained; that is, beginning at a little under the superior limit of the potential measured by the portable electrometer, it has a range of about sixteen times that of the latter. This he has called a Long-Range Electrometer. The constant of the instrument, m of (3) above, is found in the same way as that of the portable electrometer.

The attracting plate is above the guard-plate and disc as in the portable electrometer; but the former plate is fixed and the latter movable by a micrometer screw from below. The step of the screw is the same

Long Range Electrometer.

as in the other instruments, $\frac{1}{50}$ of an inch, but the range is 200 turns. The upper plate is connected with the electrified body to be tested and the guard-ring to earth, or a second body with whose potential that of the tested body is to be compared. The mode of using the instrument is thus idiostatic.

To prevent sparks the attracting plate, though plane on its lower surface, is turned over on its upper surface into a thick rim. The guard-plate is made no thicker than is necessary for stiffness, and to prevent danger of sparks between it and the upper plate projects fully an inch all round beyond the latter. No Leyden jar is required as the use is idiostatic, but a glass shade, to prevent dust and shreds which might tend to discharge the upper plate, is permanently fixed over it with an insulated electrode passing through it to the attracting plate.

Divided Ring Electrometer.

The well-nigh perfect form of Symmetrical Electrometer which we have in Thomson's Quadrant Electrometer had its beginning in the Divided Ring instrument illustrated in Fig. 52. A vertical wire carrying on one side a light horizontal needle is suspended from a fixed point. The wire passes through the centre of two flat semi-circular pieces of metal, which lie in a horizontal plane so as to form a metallic circle complete with the exception of a small space at each extremity of a diameter. These spaces insulate one semicircle from the other. Supposing the needle charged with positive electricity and made to rest in equilibrium above one of these spaces when the two semicircles are put in conducting contact, the arrangement is symmetrical about the needle. If one semi-circle be then charged with

positive the other with negative electricity, the needle will be repelled from the positive and attracted toward the negative semicircle. If then the wire be brought back and maintained in the symmetrical position by an applied couple, this couple gives a measure of that due to electric forces tending to deflect the needle, and if the potential of the needle remains constant, differences of potential established between the semicircles can be compared. Divided Ring Electrometer.

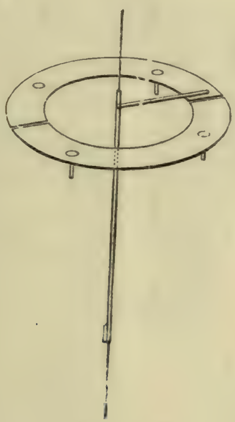

FIG. 52.

It was an obvious but important step to convert the two semicircles into four quadrants by a pair of openings along a diameter at right-angles to the other pair, to put each pair of opposite quadrants into conducting contact, and to make the needle symmetrical about the suspension wire. Thus supposing one pair of quadrants to be charged positively and the other pair negatively, Quadrant Electrometer.

Quadrant Electrometer.

one end of the needle is attracted by one of a pair of quadrants, and repelled by the adjacent quadrant of the other pair. The other end of the needle is attracted

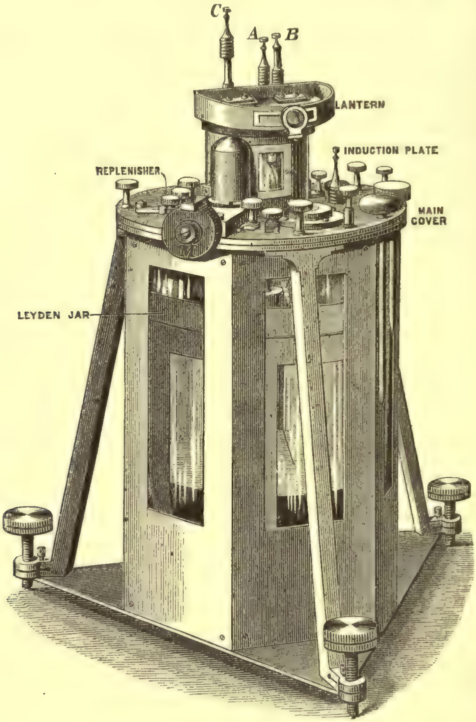

FIG. 53.

by the remaining quadrant of the first pair, and repelled by the remaining quadrant of the other pair, which is adjacent. These actions conspire to give a couple turning the needle about the suspension wire.

Final Form of Quadrant Electrometer.

In the final form of the quadrant electrometer, which is represented in Fig. 53, the four quadrants of the flat-ring are replaced by four quadrants of a flat cylindrical box made of brass. These are shown separately in Fig. 54. Each quadrant is supported on a glass stem projecting downwards from a brass plate which forms the cover of a Leyden jar, within which the quadrants and needle are enclosed. For three of the quadrants the stem passes through a slot in the cover and is attached to a brass piece which closes the slot from

Arrangement of Quadrants.

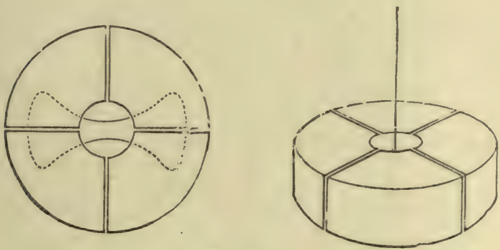

FIG. 54.

above. Thus each of the quadrants can be moved out or in through a small space. The stem of the fourth quadrant is attached to a piece above the cover which rests on three feet. Two of these feet are kept by a spring in a V groove, parallel to which the piece carrying the quadrant with it can be moved by a micrometer-screw turning in a nut fixed to the movable piece. The spring which keeps the feet of the movable piece in their groove presses outwards as well as downwards, and so keeps the same sides of the nut and

Arrangement of Quadrants.

screw threads in contact, to the prevention of "lost time." The details of the instrument will be easily made out by means of Figs. 55 and 56. The former

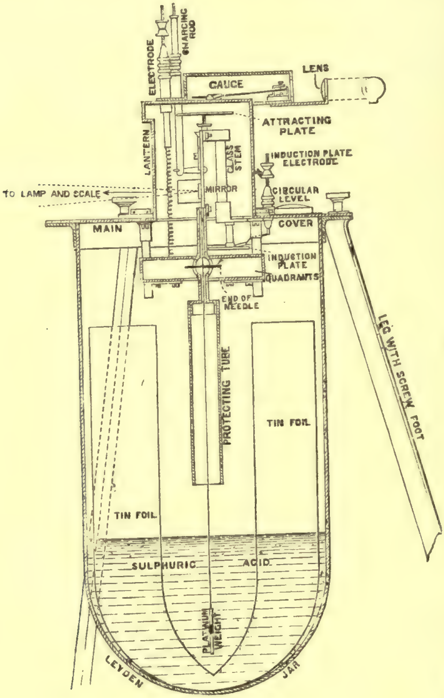

FIG. 55.

shows a vertical section of the instrument, the latter the suspension-piece and mirror.

A plate rather less in area than the upper surface of a quadrant, but of nearly the same shape, is supported

by a glass stem from the cover above a quadrant adjacent to that attached to the micrometer, and is furnished with an insulated electrode passing through the cover. Sufficient length is given to the insulating stem by attaching it to the roof of a cylinder, closed at the top, erected over an opening in the cover. The Induction Plate.

Within the box formed by the quadrants and about midway between the top and bottom, a needle of sheet aluminium of the form shown by the line drawn, partly full, partly dotted, across the plan of the quadrants on the left in Fig. 54, is suspended horizontally from two pins *c*, *d* (Fig. 56), carried by a fixed vertical brass plate supported on a glass stem projecting above the cover of the jar. The needle is attached rigidly at its centre to the lower end of a stiff vertical wire of aluminium, which passes down through an opening in the middle of the cover. The Needle and its Suspension.

To the extremities of a small cross-bar at the top of the aluminium wire are attached the lower threads of a bifilar made of two single silk-fibres. The upper ends of these fibres are wound in opposite directions round the pins *c*, *d*, each of which has, in its outer end, a square hole to receive a small key, by which it can be turned round in its socket so as to wind up or let down the fibre. By this means the fibres can be adjusted so as to be as nearly as may be of the same length; and as the whole supported mass of needle, &c., is then symmetrical about the line midway between the fibres, each bears half the whole weight. The pins *c*, *d*, are carried by the upper ends *e*, *f*, of two spring pieces which form the continuations of a lower plate Details of the Suspension.

Details of the Suspension.

screwed firmly to the supporting piece. Through *e*, *f*, and working in them, pass two screws *a* and *b*, the points of which bear on the brass supporting plate behind. By the screw *a* the end *e* of the plate *e*, *f*, can be moved forward or back through a certain range, and

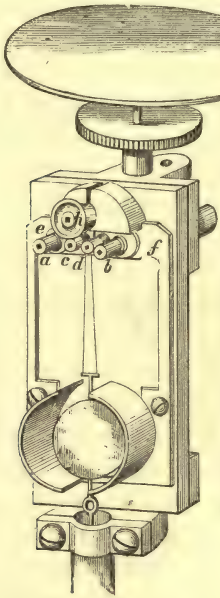

Fig. 56.

thus the pin *c* carried forward or back relatively to *d* similarly *d* can be moved by the screw *b*. Thus the position of the needle in azimuth can be adjusted. The distance of the fibres apart can be changed by

screwing out, or in, a conical plug shown between the springs *e*, *f*. Details of the Suspension.

The aluminium wire carries between its upper end and the needle a small concave mirror of silvered glass, to be used with a lamp and scale to show the position of the needle. The mirror is guarded against external electric influence by two projecting brass pieces, which form nearly a complete cylinder round it. The part of the wire just above the needle is protected by the tube shown at the bottom of Fig. 56. This tube extends down below the needle a little distance, and is cut away at each side to allow the needle free play to turn round.

The Leyden Jar.

The interior coating of the Leyden jar is formed by a quantity of sulphuric acid which it contains, and which also serves to preserve a dry atmosphere within the jar, the exterior coating by strips of tinfoil pasted on its outer surface. The acid has been boiled with sulphate of ammonia to free it from volatile impurities which might attack the metal parts of the instrument. The jar itself is enclosed within a strong metal case of octagonal form, supported on three feet, with levelling screws. The line joining two of these feet (which are in front) is, when level, parallel to the axis of the needle if the latter is properly adjusted.

The needle is connected with the inner coating of the jar by a thin platinum wire kept stretched by a platinum weight at its lower end, which hangs in the acid. The wire is protected from electrical influence by a guard-tube forming a continuation of the narrower guard-tube, partly shown in Fig. 56, and therefore

extending from below the quadrants to a short distance above the acid, and connected also by a platinum wire with the acid.

The Subsidiary Gauge-Electrometer.

The supporting plate in Fig. 56 carries the disc of an idiostatic gauge of the kind described in p. 269 above. The height of the disc is adjustable by means of a fine screw and jam-nut below it. The supporting plate, with the suspension and disc of the gauge, &c., is enclosed within an upper brass case, called the *lantern*, which closes tightly the central opening of the cover. The top of the lantern is the guard-plate of the gauge, and carries the aluminium trap-door and lever with sighting plate and lens as already described.

A glass window in the lantern allows light to pass to the mirror, and the suspension to be seen. A small opening in the glass, closed when not in use by a screw-plug of vulcanite, enables the operator to adjust the suspension without removing the lantern.

Electrodes, &c.

The principal electrodes of the quadrants are brass rods cased in vulcanite, and are arranged so as to be movable vertically. Each is terminated above in a small brass binding screw, and is connected below by a light spiral spring of platinum with a platinized contact piece, which rests by its own weight on a part of the upper surface of the quadrant, also platinized to ensure good contact. They are placed one on each side and in front of the mirror. One is in contact with the quadrant connected below to the micrometer quadrant, the other to the quadrant connected to that below the induction plate.

An insulated charging-rod descends through the lantern, and carries at its lower end a projecting spring of brass. When the rod is not in use the spring is not in contact with anything; but when the jar is to be charged the rod is turned round until the spring is brought into contact with the supporting-plate, which, as stated above, is in contact with the acid of the jar. Electrodes, &c.

The potential of the jar is maintained constant by a replenisher in the manner already described for the absolute electrometer. A spring catch keeps the knob of the replenisher, which is on the upper side of the cover, in such a position when not in use that the carriers are not in contact with any of the springs.

On the upper side of the cover are screws, three in number, by which the cover is secured to a tightly fitting flat ring collar below it, to which the jar is cemented, and to which the case is screwed; two screws, one on each side, which fix the lantern in its place; a cap covering an orifice communicating with the interior of the jar; two binding screws by which wires can be connected to the case; and a knob similar to that of the replenisher, which, when turned against a stop marked "contact," connects by an interior spring the quadrant below the induction plate with the case, and when turned in the opposite direction to an adjoining stop marked "no contact," insulates that quadrant from the case. Two keys, for turning the pins *a*, *b*, *c*, &c., are kept let down outside the case through holes in the projecting edge of the cover. The cover also carries a small circular level, set so as to have its bubble at the centre when the cover is levelled by an ordinary level.

When this has been done the accuracy of construction of the quadrants ensures that they are also level. The level has a slightly convex bottom, and is screwed down with three screws, so that when the instrument is set up for use, a final adjustment, to show horizontality of the quadrants, can easily be made by turning the screws.

Adjustments of the Quadrant-Electrometer :—Adjustment of the Needle.

Full instructions for setting up and adjusting the quadrant electrometer are sent out with each instrument by the maker, and are therefore available, if kept, as they ought to be, beside it in the case. We shall suppose therefore that the detached parts have been put into their places, the acid poured into the jar, and the instrument set up and levelled; but as a quadrant electrometer is now part of every well-equipped physical laboratory, and is used over a wide range of electrical work, we shall describe here the principal adjustments.

The two front quadrants are pulled out as far as possible, to allow the operator to observe the position of the needle, which should rest with its plane horizontal and midway between the upper and under surfaces of the quadrants. If it requires to be raised or lowered, the operator winds or unwinds the fibres by turning the pins *c*, *d*, to which they are attached. The suspension wire of the needle should pass through the centre of the circular orifice formed in the upper surface of the quadrants, when these are symmetrically arranged. If the wire is not in this position the pins *a*, *b*, are turned so as to carry the point of suspension forward or back until the wire is adjusted, and then one pin is carried

forward and the other back, without altering the position of the wire, until the black line along the needle is parallel to the transverse slit separating the quadrants.

Adjustment of the Scale.

The scale is placed at the proper distance to give a distinct image of the wire across the line of divisions in front of the lamp flame, then levelled and adjusted so that, when the image is at rest in the centre, the extremities of the scale are at equal distances from the needle.

"Hole, Slot, and Plane," for fixing position of an Instrument.

When the best relative positions of the instrument and the stand for the lamp and scale have been ascertained, these are fixed by the "hole, slot, and plane" arrangement adopted by Sir William Thomson, to allow any instrument supported on three feet or levelling screws to be removed at pleasure, and replaced without readjustment in its original position. A conical hollow, or better, a hole shaped like an inverted triangular pyramid, is cut in the table so as to receive the point (which should be well rounded) of one of the levelling screws, without allowing it to touch the bottom. A V-groove, with its axis in line with the hollow, is cut for the rounded point of another levelling screw, and the third rests on the plane surface of the table. If it is desired to insulate the electrometer case it is supported on three blocks of vulcanite cemented to the table; and in one of these the hollow is cut, in another the V-groove.

Method of charging the Electrometer-jar.

When the jar is being charged, the main electrodes, the induction plate electrode, and one of the binding screws on the cover, are kept connected by a piece of fine brass or copper wire. The charging electrode is

Method of charging the Electrometer-jar.

turned round so as to bring the spring at its lower end into contact with the supporting brass piece, and a positive charge is given to the jar by means of the small electrophorus which accompanies the instrument. The cover of the jar is tapped during the process to release the balance lever from the stop, to which it may be adhering. When the lever rises the charging rod is turned so as to disconnect the spring, and the charge is then adjusted to the normal amount (detėrmined by the distance of the attracting disc from the trap door) by the replenisher.

The spot of light may in the process of charging have moved from its position for no electrification, and must be brought back by moving out or in the quadrant carried by the micrometer screw.

Leakage of Jar: Causes and Remedies.

In ordinary circumstances the leakage of the jar will cause the hair to fall down in twenty-four hours about half the breadth of the lower black spot. This loss of charge from the jar is made good by the replenisher; but if the leakage is considerably greater, the main stem should be washed by means of a piece of hard silk ribbon (to avoid shreds) with soap and water, then with clean water, and finally carefully dried. Shreds and dust on the needle and quadrants may tend to discharge the jar, and anything of this kind should be removed by carefully and lightly dusting the needle and quadrants with a clean camel's hair brush. The jar is selected for its high insulating power, but if the acid has in careless handling of the instrument been splashed over the interior surface there may be considerable leakage over the surface of the jar to the case. This can be

remedied by removing the acid and carefully washing the jar. The replenisher may also cause leakage of the jar through a deterioration of insulating power of the vulcanite sole-plate which connects the inductors. Such a deterioration with lapse of time is not uncommon in ebonite, and is a consequence of slow chemical action at the surface. A nearly complete cure can be effected by removing the piece and washing it carefully by prolonged immersion in boiling water, and then re-covering its surface with a film of paraffin.

Leakage of Jar: Causes and Remedies.

The insulation of the quadrants is now tested. One pair of quadrants is connected to the case and a charge producing a difference of potentials exceeding the greatest to be used in the experiments is given to the insulated pair by means of a battery, one electrode of which is connected to the electrometer case, while the other is connected for an instant to the electrode of the insulated quadrants; and the deflection of the spot of light is read off. The percentage fall of potentials produced in thirty minutes or an hour is obtained merely by taking the ratio of the diminution of deflection which has taken place in the interval to the original deflection. If this is inappreciable the quadrants insulate satisfactorily. In any case, for satisfactory working the rate of loss of potential shown by the instrument should not be greater than that of the body tested.

Method of Testing Insulation of Quadrants.

If the insulation is imperfect the glass stems supporting the quadrants should be washed by passing a piece of hard silk ribbon well moistened and soaped, then with clean water to remove the soap, and dried by the same piece of ribbon well dried and warmed. If this

Remedy for Leakage.

does not succeed, the fault probably lies in the vulcanite insulators of the electrodes, which should be well steeped in boiling water, then re-covered with clean paraffin and replaced. Care must be taken if this is done not to bend the electrodes.

Adjustment of Tension of Threads.

The final adjustment of the tension of the threads to equality is now made. One pair of quadrants is connected to the case, and the other pair insulated. The poles of a single Daniell's cell are then connected to the electrodes, and the extreme range of deflection produced by reversing the battery, either by hand or by a convenient reversing key, is observed. One side of the instrument is then raised by screwing up that side by one or two turns of one of the front pair of levelling screws, and the range of deflection again noted. If the range is greater the fibre on that side is too short, if the range is smaller the fibre is too long (see p. 245 above); and the length must be corrected by turning one or other of the pins to which the fibres are suspended. The pins can be reached by the aperture in the window of the lantern ordinarily closed by the vulcanite plug; and to prevent discharge of the jar the key with vulcanite handle should be used to turn them. The black line on the needle will require readjustment by the screws after each alteration of the suspension.

Heterostatic use of the Instru-

The ordinary method of using the quadrant electrometer is heterostatic, since the jar is kept at a constant potential, generally much higher than any potential which the instrument is used to measure. The shape of the needle is such that for most practical purposes equation (65) (p. 61 above) may be regarded as giving

accurately the couple deflecting the needle, when the quadrants are symmetrical about the needle and close. Hence for small deflections we have, as in (66), for the deflection D, in terms of the potentials V, V_1, V_2 of the needle and the two pairs of quadrants respectively, the equation

Heterostatic use of the Instrument.

$$D = c(V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right) \quad . \quad . \quad (4)$$

where c is a constant depending on the instrument and the mode of reckoning of D. If V be, as it usually is, great in comparison with V_1 or V_2, then

$$V_1 - V_2 = C'D \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where C' is the now practically constant value of $c\{V - (V_1 + V_2)/2\}$.

If the angle of deflection θ of the ray of light is not a very small angle, the couple given by the bifilar, it is to be remembered, is proportional to $\sin \frac{1}{4}\theta$. Hence if D be the distance in divisions on the scale (supposed straight and at right angles to the zero direction of the ray) through which the spot of light is deflected, and R the horizontal distance of the scale from the mirror in the same divisions, we have $\tan \theta = D/R$, from which θ can be found and hence $\frac{1}{4}\theta$. We have then

$$K \sin \tfrac{1}{4}\theta = (V_1 - V_2)\left(V - \frac{V_1 + V_2}{2}\right)$$

where K is a constant.

Equation (4) would be more nearly satisfied if the central portions of the needle to well within the quadrants were as much as possible cut away, leaving only a framework opposite the orifice at the centre of the quadrants to support the needle.

Grades of Sensitiveness.

The electrometer, when used heterostatically, admits of a number of different grades of sensibility. These are shown in the two following tables, where L denotes the electrode of the pair of quadrants, one of which is below the induction plate, R the electrode of the other pair of quadrants, I the electrode of the induction-plate, O an electrode of the case of the instrument, and C the electrode of the conductor to be tested. LC denotes that L is connected to C, RO that R is connected to O, RLC that RL and C are connected together, and so on, (L) that the quadrants connected with L are insulated by raising L, (R) that the quadrants connected with R are similarly insulated, (RL) that both L and R are raised. The disinsulator mentioned (p. 291 above) is used to free the quadrants connected with L from the induced charge which they generally receive when L is raised.

GRADES OF SENSITIVENESS.

A.

Inductor connected with quadrant beneath it.

FULL POWER

$$\begin{bmatrix} LC \\ RO \end{bmatrix} \text{ or } \begin{bmatrix} RC \\ LO \end{bmatrix}$$

DIMINISHED POWER.

$$(L) \begin{bmatrix} RC \\ O \end{bmatrix} \text{ or } (R) \begin{bmatrix} LC \\ O \end{bmatrix}$$

B.

Inductor connected as indicated below.

FULL POWER

Inductor Insulated.

$$\begin{bmatrix} LC \\ RO \end{bmatrix} \text{ or } \begin{bmatrix} RC \\ LO \end{bmatrix}$$

GRADES OF DIMINISHED POWER.

$$(L) \left[\begin{matrix} \begin{bmatrix} RC \\ IO \end{bmatrix} \\ \begin{bmatrix} RIC \\ O \end{bmatrix} \\ \begin{bmatrix} IC \\ RO \end{bmatrix} \end{matrix}\right] \text{ or } (R) \left[\begin{matrix} \begin{bmatrix} LC \\ IO \end{bmatrix} \\ \begin{bmatrix} LIC \\ O \end{bmatrix} \\ \begin{bmatrix} IC \\ LO \end{bmatrix} \end{matrix}\right]$$

$$(RL) \begin{bmatrix} IC \\ O \end{bmatrix}$$

Either of these grades of sensibility may of course also be varied by increasing the distance of the fibres apart.

Graduation of Quadrant Electrometer.

The quadrant electrometer can be made to give results in absolute measure by determining the constant C' of equation (50), by which the deflection must be multiplied to give the difference $V_1 - V_2$. This can be done by observing the deflection produced by a battery of electromotive force of convenient amount, determined by direct measurement with an absolute electrometer or otherwise. Different such electromotive forces may be employed to give deflections of different amounts and thus give a kind of calibration of the scale to avoid error from non-fulfilment of condition of proportionality of deflection to difference of potentials.

Idiostatic Use of Quadrant Electrometer.

The quadrant electrometer may also be used idiostatically for the measurement of differences of potential of not less than about 30 volts. The volt is the practical unit of electromotive force, and is about 1·07 times the electromotive force of a Daniell's cell. For its definition see Vol. II. When it is so used the jar is left uncharged, the charging-rod is brought into contact with the inner coating of the jar, and joined by a wire with one of the main electrodes, so as to connect the needle to one pair of quadrants. The other pair of quadrants is either insulated or connected to the case of the instrument. The instrument thus becomes a condenser, one plate of which is movable, and by its change of position alters the electrostatic capacity of the condenser. The two main electrodes are connected with the conductors, the difference of potentials between which it is desired to measure.

Idiostatic Use of Quadrant Electrometer.

A lower grade of sensibility can be obtained by connecting the needle through the charging-rod to the electrode R, and using the induction-plate instead of the pair of quadrants connected with L, which are insulated by raising their electrode.

When the instrument is thus used idiostatically V in equation (49) above becomes equal to V_1, and instead of (50) we have

$$D = \frac{C}{2}(V_1 - V_2)^2 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

that is, the deflection is proportional to the square of the difference of potentials and therefore independent of the sign of that difference. It is to the left or right according to the electrode connected to the needle. This independence of sign in the deflection renders the instrument thus used applicable to the determination of potentials in the circuits of alternating dynamo- or magneto-electric generators. (See below, Vol. II.)

Modifications of Quadrant Electrometer.

The quadrant electrometer has been modified by different makers. In a form made in Paris for M. Mascart, the needle is kept at a constant potential by being connected to the positive pole of a dry pile, the negative pole of which is connected to the case, and the replenisher is dispensed with.

In another form devised by Prof. Edelmann of Munich, and suitable for some purposes as a lecture-room instrument, the quadrants are longitudinal segments of a somewhat long vertical cylinder, and the needle consists of two coaxial cylindric bars connected by a cross-frame, and suspended by means of a bifilar.

A glass vessel below contains strong sulphuric acid in which dips a vane carried by a platinum wire attached to the needle.

Sir W. Thomson's Electrostatic Voltmeter.

For practical work Sir William Thomson has lately constructed a form of electrometer to be used idiostatically, and has called it an electrostatic voltmeter. It is represented in Fig. 57, and may be described as an air condenser, one plate of which, corresponding to the needle of the quadrant electrometer, is pivoted on a horizontal knife-edge working on the bottoms of rounded V-grooves cut in the supporting pieces. This plate by its motion alters the electrostatic capacity of the condenser. The fixed plate consists of two brass plates in metallic connection, each of the form of a double sector of a circle, which are placed accurately parallel to one another, with the movable plate between them as shown in the figure. The upper end of the movable plate is prolonged by a fine pointer which moves along a circular scale, the centre of which is in the axis. The fixed plates are insulated from the case of the instrument; the needle is uninsulated.

Contact is made with the plates by insulated terminals fixed outside the case. The two shown on the left-hand side in the figure belong to the fixed plate, and a similar pair on the right-hand side are in connection with the movable plate through the supporting V-groove and knife-edge. The terminals of each pair are connected by a safety arc of fine copper wire contained within a U-shaped glass tube suspended from the terminals, and the terminals in front in the diagram which are separated from the plates by the arcs of wire

Sir W. Thomson's Electrostatic Voltmeter.

are alone used for connecting to the conductors, or two points of an electric circuit, the difference of potentials

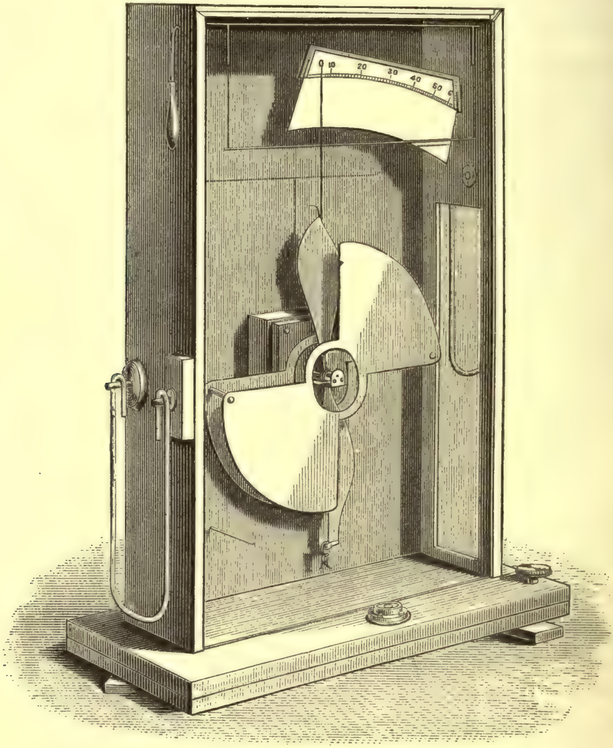

Fig. 57.

between which is to be measured, and are therefore called the working terminals.

When a difference of potentials is established between the fixed and movable plates the latter move so as to increase the electrostatic capacity of the condenser, and the couple acting on the movable plate in any given position is, as in the quadrant electrometer when used idiostatically, proportional to the square of the difference of potentials. This couple is balanced by that due to a small weight hung on the knife-edge at the lower end of the movable plate. Sir W. Thomson's Electrostatic Voltmeter.

The scale is graduated from 0° to 60° so that the successive divisions represent equal differences of potential. Three different weights, 32·5, 97·5, 390 milligrammes respectively, are sent with the instrument to provide for three different grades of sensibility. Thus the sensibility with the smallest weight on the knife-edge is a deflection of one division per 50 volts, with the two smaller weights, that is four times the smallest, one division per 100 volts, with all three weights or sixteen times the smallest weight, one division per 200 volts. Scale of Voltmeter.

The electrostatic voltmeter is graduated as follows. A known difference of potentials is obtained by means of a battery of from 50 to 100 cells with a high standard resistance in its circuit. An absolute galvanometer or current balance (see Vol. II.) measures the current in the circuit, and the product of the numerics of the current and the resistance gives that of the potential-difference between the terminals of the latter. These terminals are connected to the working terminals of the voltmeter, and the deflections noted with the smaller weights on the knife-edge. Graduation of Voltmeter.

Graduation of Voltmeter.

For the higher potentials a number of condensers of good insulation are joined in series and charged by an application of the wires from the terminals of the resistance coil to each condenser in succession from one end of the series to the other. This is done so as to charge each condenser in the series in the same direction, and as the same difference of potentials, V say, is produced between the plates of each condenser, the total difference between the extreme plates is nV, if there be n condensers. A convenient large potential-difference can thus be obtained with sufficient accuracy, and being applied to the working terminals of the voltmeter is made to give divisions for a series of different weights hung on the knife-edge. These divisions correspond of course to deflections for known potentials with *one* of the weights on the knife-edge.

The divisions thus obtained are then checked by using three instruments which have been dealt with in this way. They are joined in series and a difference of potentials established between the extreme terminals, which is observed also by the third joined across the other two. Thus by a process of successive halving and doubling the scale is filled up.

A description of Lippmann's capillary electrometer will be given in Vol. II. in connection with the Measurement of Electromotive Forces.

CHAPTER VI.

THE COMPARISON OF RESISTANCES.

We give here some account of methods for the comparison of the resistances of conductors in which steady currents are kept flowing. In most cases the conductor to be compared is arranged in a particular way in connection with other conductors, which are then adjusted so as to render the current through a certain conductor of the system zero. From the known relation of the resistances of the other conductors the required comparison is deduced. In this and in other arrangements the existence of an electric current has to be observed, and in some cases the amount of the current must be measured. It is therefore necessary, although the subject of the measurement of currents belongs properly to the electromagnetic part of this work, to describe shortly the means adopted in the comparison of resistances to detect, and, when required, to compare currents of electricity.

The instrument used is called a *galvanometer*. Its action is based on the phenomenon observed by Oersted and explained by the electromagnetic theory of Ampère, that if a wire, along which a current is flowing, be held parallel to a magnetic needle resting in equilibrium

Oersted's Discovery. Principle of Galvanometer.

Oersted's Discovery. Principle of Galvanometer.

under the action of magnetic force, it will be deflected from its original position towards a position at right angles to the wire. To fix the ideas, let the needle be a thin straight longitudinally magnetized bar, in equilibrium under the action of magnetic force with its length horizontal, and free to turn round a vertical axis; and let the wire carrying the current be stretched parallel and near to the needle above it or below it. The direction in which the needle turns round is reversed if the wire, supposed first placed above the needle, is then placed below it: again it is reversed when the portion of wire held near the needle is turned end for end without other change of position. An augmented deflection is therefore obtained if the wire is bent round so that one portion is above and the other below the needle, and a still greater when the wire supposed covered with non-conducting material, is wound closely into a coil of several turns which is then placed with its plane parallel to the length of the needle; for the effects due to the upper and to the lower portions of the wire are then in the same direction. A coil of wire thus placed relatively to a magnetic needle suspended so as to be free to turn round a fixed (generally vertical) axis is a galvanometer.

Testing-Galvanometers.

Galvanometers for ordinary testing purposes are generally made by winding wire covered with silk, or some other non-conducting substance, round a hollow core or bobbin, symmetrical about a straight axis. A magnetic needle, generally composed of two or three or more small, and as nearly as may be equal magnets, relatively fixed parallel to one another with their like

Testing-Galvanometers.

poles turned in the same direction, and their centres in a plane perpendicular to their lengths, is suspended with its centre * at some convenient point (generally the middle point) of the axis of the coil. The coil is so placed and levelled that the needle, supposed at rest under the action only of the magnetic force of the field in which the apparatus is placed, has its length at right angles to the same axis.

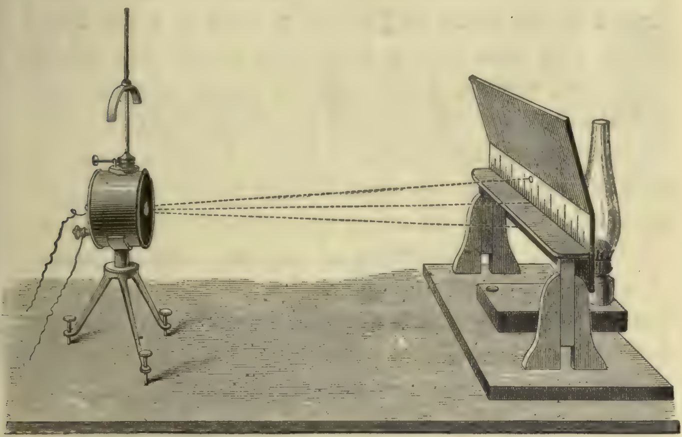

Fig. 58.

The form of galvanometer generally employed in the measurement of resistances is the reflecting galvano-

* The magnetic axis and magnetic centre of such an assemblage of magnets will be defined in the Chapter on Magnetism ; for our present purpose it is sufficient to say that we can adjust the galvanometer so that the magnetic axis and centre may be as nearly as we please in any required position.

Thomson's Reflecting Galvanometer.

meter invented by Sir William Thomson, one arrangement of which is shown in Fig. 58. For most purposes the ordinary form of the instrument can be used. In this a mirror of silvered glass to which the needle-magnets are cemented at the back is hung within a cylindrical cell about half a centimetre in diameter. The ends of the cylinder are closed by glass plates from four to five millimetres apart, held in brass rings which can be screwed out or in so as to increase or diminish the length of the cell. The mirror is hung by a piece of a single silk fibre passed through a small hole in the cylindrical surface of the chamber and fixed there with a little shellac. The mirror is only of slightly smaller diameter than the cylinder in which it hangs, so that in this arrangement the fibre is very short, rendering it necessary in cases in which deflections have to be read off to allow for the effects of torsion. The cylindrical chamber is screwed into one end of a cylinder of slightly greater diameter which fits the hollow arc of the coil, and is called the galvanometer-plug. When the plug is in position the mirror hangs freely within its cell, with therefore the point of suspension on the highest generating line of the cylinder.

Method of observing Deflections.

Deflections of the needle are observed either by the Poggendorff telescope method, or, and much more generally, by the ordinary projection method described on p. 211 above.

Dead-Beat Galvanometer.

The weight of the needle and mirror is under one grain, and hence the period of free vibration of the suspended system about any position of equilibrium is short. The needle is also made to come quickly to rest by the smallness of the chamber in which it hangs. Since the mirror

nearly fills the whole cross-section of the cell, the air damps the motion of the mirror to a very great extent even when the cell has its largest volume. The mirror may be made quite "dead-beat" (p. 224 above) by screwing in the front and back of the cell until the space is sufficiently limited.

Dead-Beat Galvanometer with Long Fibre.

In instruments in which it is desirable to avoid effects of torsion the galvanometer coil is made in two lengths, which are fixed end to end, with a narrow space between them to receive the suspension piece. This piece forms a chamber in which the needle hangs between the two halves of the coil and gives a length of fibre which at shortest is equal to the radius of the outer case of the coil, and which can obviously be made as long as is desired. The part of the hollow core at the needle is closed in front and at back by glass plates carried by brass rings. These can be screwed in or out by a key from without so as to diminish or increase the size of chamber, and thus render the needle system more or less nearly "dead-beat."

Process of setting up a Galvanometer.

The galvanometer is generally set up so that the deflections are read by the ordinary deflection method (p. 211 above). It is only necessary to arrange that the needles when no current is flowing in the wires shall hang parallel to the plane of the coils. This is done as follows. A straight thin knitting wire of steel is magnetized and hung by a single silk fibre of a foot or so in length. This can easily be done by taking a sufficiently long single fibre of silk and forming a double loop on one end by doubling twice and knotting. In this double loop, made widely divergent, the steel wire is laid

Process of setting up a Galvanometer.

horizontally, and the single end of the fibre is attached to a support carried by a convenient stand, which is then placed so that the wire takes up a position in the direction of the horizontal component of the magnetic field where the needle is to be placed. A line can now be drawn parallel to the wire on the table beneath it. All that is necessary then is to place the galvanometer so that the front and back planes of the coil are vertical and parallel to this line, and adjust the lamp and scale as described above.

It is sufficient for our present purpose to state that if the needles be so small as in the Thomson reflecting galvanometer, and torsion can be neglected, the current in the coil may be taken as proportional to the tangent of the deflection angle, and therefore if that angle be not greater than three or four degrees the current may, with an error not greater than $\frac{1}{7}$ per cent., be taken as proportional to the deflection simply. We shall discuss the measurement of currents fully in later chapters.

Adjustment of Field for sensibility.

The galvanometer should be made as sensitive as possible by diminishing the directive force on the needle as far as is practicable without rendering the needle unstable. This is easily done by placing magnets near the coil so that the needle hangs, when the current in the coil is zero, in a very weak magnetic field. That the field has been weakened by any change in the disposition of the magnets, made in the course of the adjustment, will be shown by a lengthening of the period of free vibration of the needle when deflected for an instant by a magnet and allowed to return to zero. The limit of instability has been reached when the position of the

spot of light for zero current changes from place to place on the scale, and the intensity of the field must then be slightly raised to make the zero position of the needle one of stable equilibrium.

Adjustment of Field for sensibility.

Although not absolutely essential, except when accurate readings of deflections are required, it is always well, when the field is produced by magnets, to arrange them so that the field at the needle is nearly uniform. It may therefore be produced by two or more long magnets placed parallel to one another at a little distance apart symmetrically with respect to the centre of the needle above or below it, and with their like poles turned in the same directions; or a long magnet placed horizontally with its centre over the needle, and mounted on a vertical rod so that it can be slided up or down to give the required sensibility, may be used.

Sensibility is sometimes obtained by the use of astatic galvanometers, but these are rarely necessary and are more troublesome to use than the ordinary non-astatic instrument. Such galvanometers will be described in Vol. II.

Resistance Coils.

For the comparison of the resistances of conductors other resistances the relations of which are known are employed. These are generally coils of insulated wire wound on bobbins which are arranged so that the coils can be used conveniently in any desired combination. Such an arrangement of coils is called a resistance box. Figs. 59 and 60 show resistance boxes of different forms.

Construction of a Resistance Box.

In a resistance box each coil has a separate core, which ought to be a brass or copper cylinder split longi-

Construction of a Resistance Box.

tudinally to prevent induction currents, and covered with thin rubber or varnished paper for insulation. These cores are shown in Fig. 61. The metallic core facilitates the cooling of the coil if an appreciable rise of temperature is produced by the passage of a current through it. After each layer of the coil has been wound it is dipped in melted paraffin, so as to fix the spires relatively to one another, preserve them from damp, and insure better insulation. It is of

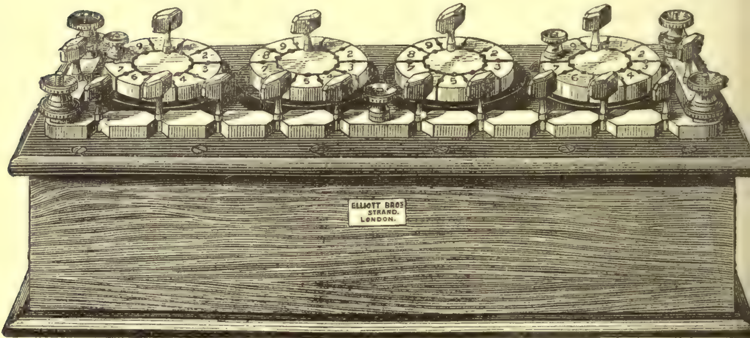

Fig. 59.

great importance to use perfectly pure paraffin, and especially to make sure that no sulphuric acid is present in it. Unless this precaution is observed trouble may be caused not only by the action of the acid on the metal of the conductor, but by the polarization effects due to electrolytic action in the acid paraffin. Paraffin which is at all doubtful should be well shaken up when melted with hot water, to remove the acid.

The wire chosen for the higher resistances is generally an alloy of one part platinum to two parts silver. This has a high specific resistance (p. 380 below), combined with a small variation of resistance with temperature. For the lower resistances wire of greater thickness is employed on account of its greater conductivity, which enables a greater length of wire to be used and thus facilitates accurate adjustment. Material of Coils.

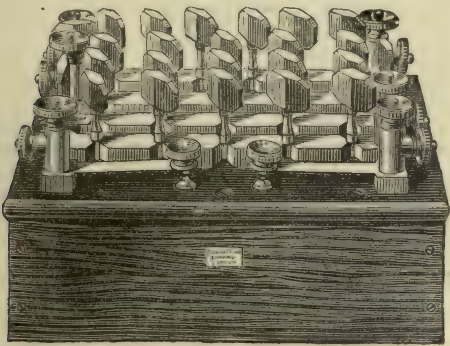

Fig. 60.

Coils are now sometimes made of "platinoid," a species of German silver which does not tarnish seriously with exposure to the air and has a low variation of resistance with temperature (see Table V.).

When a coil of given resistance is to be wound, a length of well-insulated wire of slightly greater resistance (determined by comparison at ordinary temperature by one of the processes to be described) is cut, doubled on itself at its middle point, and wound thus Winding of Coils.

Winding of Coils.

double on its core. This is done to avoid the effects of induction (see 335 below) when the current is in a state of ariation, as when starting or stopping. After the coil has been wound its resistance is again measured, and if good insulation has been obtained, it ought now to show as lightly increased resistance, on account of the change produced in the wire by bending. The coil is fixed

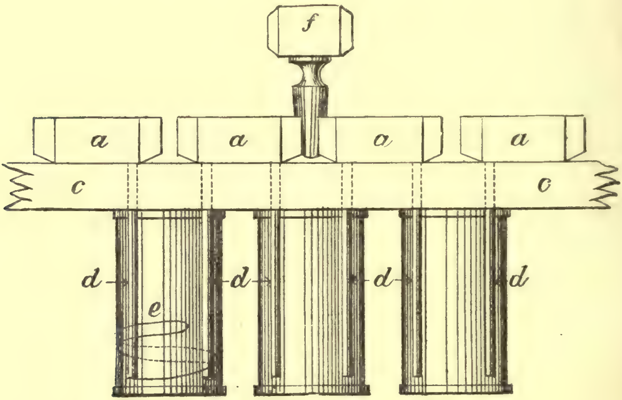

FIG. 61.

in position by two long brass or copper screws *d, d*, Fig. 61, passing through ebonite discs in the ends of its core, which fasten it to the cover of the box. These should be sufficiently massive to give no appreciable resistance. These screws are attached to two adjacent brass pieces *a, a*, on the outside of the cover, and have the ends of the wire of the coil soldered to them so that the coil bridges across the gap shown

in the figure between every adjacent pair of brass pieces. The coil is now brought to the temperature at which it is to be accurate and finally adjusted so that its resistance taken between the brass pieces is the required resistance.

The "Ohm" or Practical Unit of Resistance.

Coils are made in multiples of the "Ohm" or practical unit of resistance. The ohm will be defined absolutely in the second part of this work: it is sufficient at present to say that the *Legal Ohm* as adopted by the International Congress of Electricians held at Paris in 1884 is equal to the resistance of a uniform column of pure mercury 106 centimetres long and one square millimetre in cross-section, at the temperature 0° C. The mode of realizing such a standard is described below, p. 384.

Arrangement of Coils in Box.

A series of coils are arranged in a resistance box in some convenient order either in series or in multiple arc. Fig. 62 shows a series arrangement suitable for many purposes. The numbers indicate the number of ohms in the corresponding coils. The space between each pair of blocks is narrow above and widens out below, as shown in Fig. 62, to increase the effective distance along the vulcanite from block to block. In the adjacent ends of the brass pieces, between which is the narrow gap, are cut two narrow opposite grooves, so as to form a slightly conical vertical socket. This fits a slightly conical plug, *f* in Fig. 61, which when inserted bridges over the gap by making direct contact between the blocks, and when not thus in use is held in a hole drilled in the middle of the upper surface of the block. The coil is short-circuited when the plug is inserted, that is a current sent from one block to

the other passes almost entirely across the plug on account of the much greater resistance of the coil. The handle, *f*, of the plug is generally made of ebonite.

Box arranged in Geometrical Progression.

The plan of arranging a series resistance box which is most economical of coils is a geometrical progression with common ratio 2. In such a box two units are generally

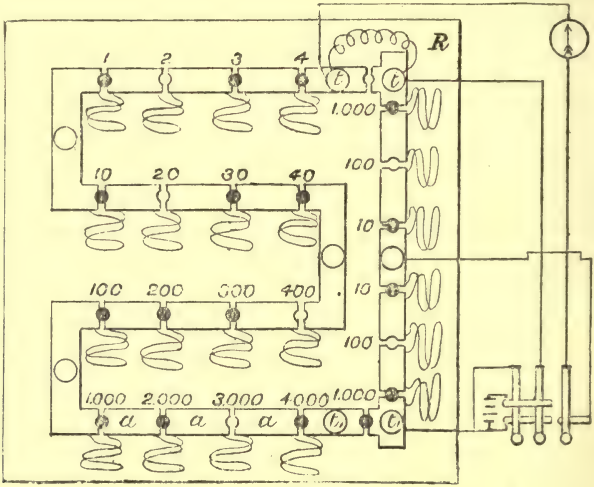

FIG. 62.

provided to enable the box to be conveniently tested. The inconvenience of the arrangement is in the reduction of any resistance which it is proposed to unplug in the box to its expression in the binary scale of notation. For example if the resistance 370 is to be found on the box, this is expressed as $2^8+2^6+2^5+2^4+2$

Box arranged in Geometrical Progression.

or 101110010, and the corresponding plugs inserted, namely the first plug beyond the units, and the fifth, sixth, eighth beyond the units. The process of reduction is performed as follows by dividing successively by 2, and writing the remainders as successive figures of the number from right to left in the order in which they are obtained, ending with the last quotient, which is of course 1.

2	370	
	185	0
	92	1
	46	0
	23	0
	11	1
	5	1
	2	1
	1	0

Hence 370 = 101110010 in the binary scale. It is not however always necessary to go through this process. Practice with a box on this principle leads soon to readiness in deciding what coils are to be unplugged, or what is the resistance of any set of coils which may be unplugged. It is well to remember that any coil of the series is greater by unity than the sum of all the preceding coils of the series.

Sir W. Thomson's Resistance Box.

Figs. 63 and 64 show the arrangement of coils in a resistance-box lately invented by Sir William Thomson, in which the geometrical progression arrangement has been adopted. The interior of the box is a copper cylinder with projecting rings soldered round it so as to

Sir W. Thomson's Resistance Box.

form recesses in which the coils are wound. The coils are made of platinoid wire well insulated with silk, and are wound double in the usual way. An outside cylinder of copper is screwed on round the rings, and

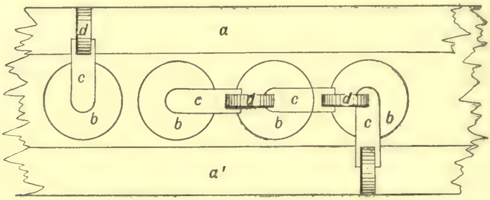

FIG. 63.

a rise of temperature in any part is rapidly equalized by the surrounding copper case.

The pieces marked *a, a,* are copper or brass plates, here shown straight, but in the actual instrument usually

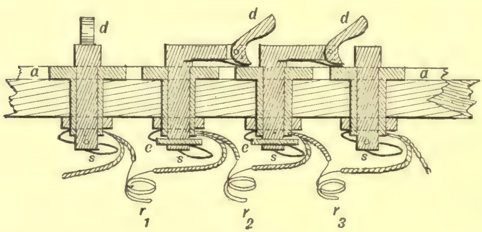

FIG. 64.

thus circular rings. Between these plates are seen discs *b, b, b,* &c., of gun metal, which carry tubes passing down through the vulcanite top of the box as shown in

the lower figure. The vertical stem of a knee-piece passes through each tube, and the horizontal part carries beneath it a spring projecting over the next disc. A coil is placed between each successive pair of tubes, and is cut out by depressing the corresponding cam lever d, which brings the spring underneath into contact with the disc over which it projects. The contact between the stem, each knee-piece, and its tube is shunted over by a flexible spiral of copper; and the piece is prevented from rising when the lever cam is depressed by a split-ring round its lower end.

Sir W. Thomson's Resistance Box.

The flexible spirals are protected from damage by over twisting by stops which prevent each knee-piece receiving more than half a turn. In the Fig. the pieces a, a are shown connected by the three resistances r_1, r_2, r_3, in series; but any combination of series and multiple arc can be obtained by turning round the knee-pieces so as to make contact with the terminal-pieces a, a, when the cam levers d are depressed. Thus by turning these pieces so as to make contact with a and a' alternately, the whole series of coils can be arranged in multiple arc.

The coils form a geometrical series from 1 to 4096 with a common ratio 2. The unit is duplicated for the reason stated above.

"Dial" form of Resistance Box.

The "Dial" form of series resistance box shown in Fig. 59 above, is preferable to the ordinary forms for many purposes. It contains three or four or more sets of equal coils, each nine in number. One set consists of nine units, the next of nine tens, the next of nine hundreds, and so on. Besides these the box

"Dial" form of Resistance Box.

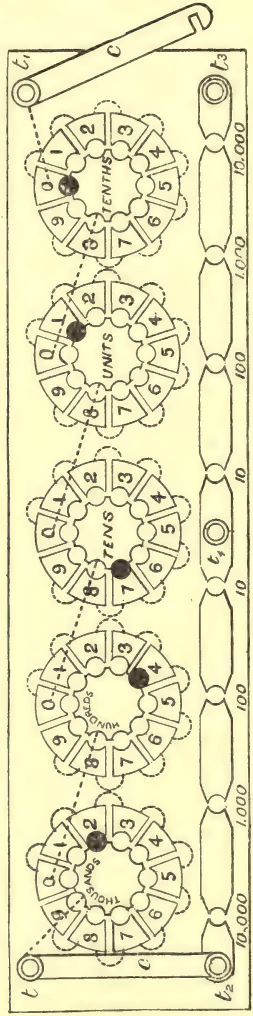

FIG. 65.

sometimes contains a set of nine coils each a tenth of a unit. Fig. 65 is a plan of a five-dial box. The sets of coils are arranged along the box in order of magnitude. Each set is arranged in series, and the blocks to which the extremities of the coils are attached are arranged in circular order round a central block, which can be connected to any one of the ten blocks of the set surrounding it, by inserting a plug in a socket provided for the purpose. Each central block, except the first and last, is connected by a thick copper bar inside to the initial block of the succeeding series of nine coils, as shown in Fig. 65 by the dotted lines. The ten blocks of each set of coils are numbered 0, 1, 2, . . . 9, as shown. Thus a current passing to one of the central blocks passes across through the bar to the next series of coils, then through the coils until it reaches a block connected to

the central piece by a plug, when it passes across to the centre and then to the next series of coils. If no coil of a series is to be put in circuit, the plug joins the central block to the coil marked zero.

Arrangement of Coils in Dial Box.

In a five-dial box the central blocks are marked respectively TENTHS, UNITS, TENS, HUNDREDS, THOUSANDS, and the resistances are read off decimally at once. Thus supposing the centre in the first dial to be connected to the block marked 5, in the second dial to the block marked 7, in the third to that marked 6, the resistance put in circuit is 67·5 units.

Advantage of Arrangement.

The advantage of the arrangement consists in the fact that only one plug is required in each dial whatever the resistance may be, and since the plugs when no coils are included complete the circuit through the zeros, there is always the same number of plug contacts in circuit, instead of a variable number as in the ordinary arrangement.

Bridge Coils.

Besides the dial resistances there is generally in each box a set of resistances arranged in the ordinary way, and comprising two tens, two hundreds, two thousands, and sometimes two ten-thousands, fitted with terminals to allow the box to be conveniently used as a Wheatstone Bridge, as described below. The extremities of this series of resistances can be connected by means of thick copper straps with the series of dial resistances. Each pair of equal coils are sometimes wound on one bobbin to ensure equality of temperature.

It is sometimes desirable to have a ready means of varying the ratio of two resistances, or of increasing a single resistance by steps of any required amount.

Resistance Slides. For this purpose a resistance slide is a convenient arrangement. A form devised by Sir William Thomson is shown at *CD* in Fig. 66. Along a metallic bar *r* in front of a series of equal resistance coils slides a contact piece *s* by which *r* is put in conducting contact with any one of the series of brass or copper blocks by which the coils are connected. The figure shows a combination of two slides used by Sir William Thomson

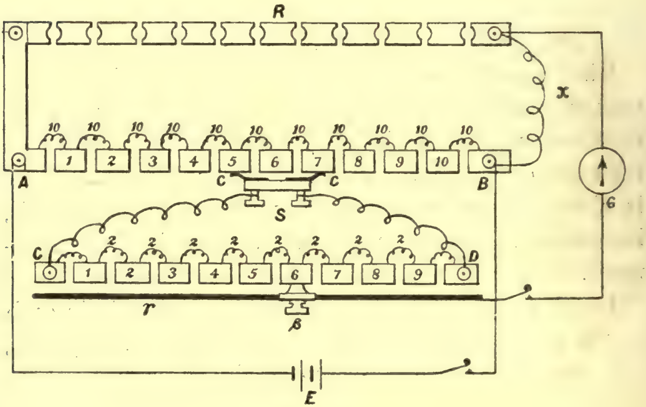

FIG. 66.

and Mr. C. F. Varley for cable testing. Each resistance in *AB* is five times that of each coil in *CD*, and there is the same number in each, so that the whole resistance of *CD* is twice that of each coil in *AB*. The slider, *S*, of *AB* consists of two contact pieces insulated from one another on the slider, and at such a distance apart as to embrace two coils. The terminals of *CD*

Vernier-Slide.

are connected to CC as shown in the figure, and therefore in whatever ratio the resistance CD is divided by the contact piece s, in that ratio is the joint resistance of the two coils CC divided. CD thus forms a vernier for AB. In the arrangement figured the resistance CD is divided into the two parts 12 and 8, and therefore the sixth and seventh coils of AB which are between the terminals of S are divided into two

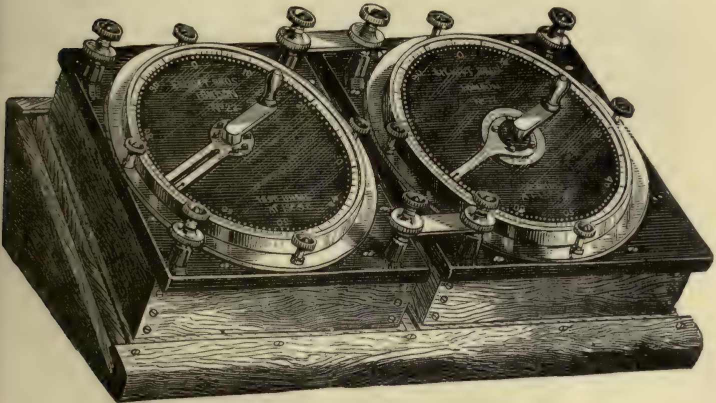

FIG. 67.

similarly situated parts 12 and 8. Hence the whole resistance between A and B is divided into the two parts 56 and 44.

Fig. 67 shows a dial form of the double resistance slide. The main coils are on the left, the vernier coils on the right. Each slide may be detached and used independently if required.

Conductance or "Mho" Boxes.

Boxes in which the coils in circuit are in multiple arc were first made at the suggestion of Sir William Thomson, and called Conductivity * Boxes, because the conductance in circuit is obtained by adding the conductances of the coils. Fig. 68 shows the arrangement. Each coil is a resistance coil wound on a bobbin as described above and has one extremity connected to a massive bar a, the other to a brass block c, outside the box, which can be connected by a plug to the

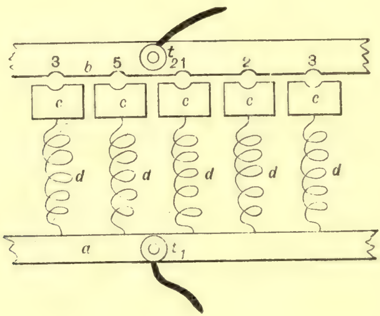

Fig. 68.

massive bar b. The resistance in circuit is obtained at once by adding the conductances of the coils thus in circuit, and taking the reciprocal of their sum. The conductances of the coils are marked on the corresponding blocks outside the cover.

This arrangement is very convenient for the measure-

* The word "Conductance" (see p. 204) is now being widely used instead of "Conductivity," and we shall in this chapter and henceforth adopt the term.

ment of low resistances such as one ohm and under, as it gives a long gradation of fractions by combination of the coils.

Sir William Thomson has proposed to call a box arranged thus a Mho-Box, where "Mho" is the word "Ohm" read backwards to indicate that the box gives conductances, that is reciprocals of resistances.

Effect of Rise of Temperature on Resistance.

The resistance of almost all wires increases with rise of temperature, and the box is generally adjusted to be correct at a convenient mean temperature which is marked on the cover. The value of the resistance shown by the box at any other temperature is obtained when the change of temperature can be ascertained from the known variation of resistance with temperature. A table of the variation of the resistances of different substances with temperature is given at the end of this volume.

The general internal temperature can be observed by means of a thermometer passed through one of the orifices which should be left in the side of the box to allow free circulation of air. Local changes of temperature may sometimes be produced in the coils without affecting appreciably the general internal temperature. These changes cannot be accounted for, as it is impossible to observe them with any accuracy, but can be avoided by using only the very feeblest currents, and continuing these for the shortest possible time.

Temperature determined by an Auxiliary Copper Coil.

The general internal temperature can also be measured by means of an auxiliary coil provided for the purpose. This is constructed of thick copper wire wound on ebonite, and extends along the whole length of the box.

Temperature determined by an Auxiliary Copper Coil.

Since the variation of resistance of copper relatively to that of the wire of which the coils are constructed is known, we can by measuring the resistance of this auxiliary unit by the box itself obtain a closely approximate estimate of the internal temperature.

The temperature variation may be made for all the coils the same as the highest variation for any one, by introducing into each a piece of copper (conveniently at the bight after the coil is wound) just sufficient for the purpose.

Process of Testing a Resistance Box.

In every case the blocks to which the coils are attached should be pierced with a socket for special plugs with binding terminals attached, by means of which any coil in the box may be brought into circuit itself. This is necessary for the testing of the box, which is done as follows. In the case of the ordinary arrangement of coils Fig. 59, each of the units is compared with a standard unit, then the two units together are tested against each of the 2s, then the 2s and a 1 are tested against the 5 and so on, until the 100s are reached. All the preceding coils put together give 100, which can be tested against each of the 100s, and this process is continued until the box is completely tested. The process can be checked by other possible combinations, and the whole of the results, if necessary, put together by the ordinary methods of combination.

If a dial box is to be tested the auxiliary unit, if it has one, suffices for the comparison of each of the units, then the nine units and the auxiliary unit give 10 for the comparison of each of the nine tens. These when compared give with the ten units

100 for the comparison of each of the hundreds, and so on.

Process of Testing a Resistance Box.

In the case of a box arranged in geometrical progression with common ratio 2, and first term 1, the unit is duplicated for the sake of comparison. Each unit having been compared with a standard, they give together a comparison of the next coil, which is 2, then that with the two units give 4, with which the coil of 4 units can be compared, and so on.

The actual methods of comparing coils are described below (p. 353 *et seq.*). It is to be remembered that in the comparison of the coils of low resistance the connecting wires (which should be in all cases short and thick) must be taken into account.

Care of Box.

In the use of a set of resistance coils it is important that the plugs be kept clean, and the ebonite top of the box, especially between the blocks of brass, kept free from dust and dirt. The ebonite may be freed from grease by washing it with benzole applied sparingly by means of a brush, and a film of paraffin oil should then be spread over its surface. The plugs and their sockets may also be freed from adhering greasy films by washing in the same way with benzole or very dilute caustic potash. The latter should not however be allowed to wet the ebonite surface. If necessary the sockets may be scraped with a round-pointed scraper. On no account should the plugs or sockets be cleaned with emery or sand paper.

It is frequently necessary to adjust a current to a convenient strength by varying the amount of resistance in circuit. When the amount of resistance in

Rheostat. circuit need not be known, this can be done most readily by means of a rheostat, or resistance coils in series with a rheostat, an arrangement which has the advantage of giving a continuous variation of the resistance. A form of rheostat constructed by Sir William Thomson is shown in Fig. 69. Two metal

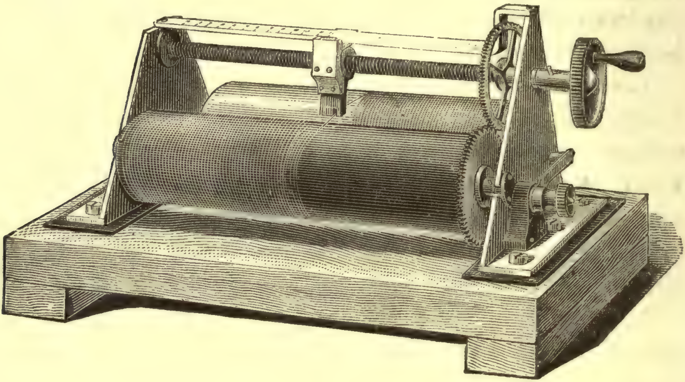

FIG. 69.

cylinders are mounted side by side on parallel axes and are geared so as to be driven at the same rate in the same direction by a third shaft turned by a crank. Along this shaft from end to end of the cylinder is worked a screw, which when turned moves a nut along a graduated scale at the top of the instrument. One of the cylinders is covered with well varnished paper, the other has a clean metal surface. A bare wire of platinoid or other material is wound partly on one cylinder partly on the other, and in passing from one

cylinder to the other threads through a hole in the nut. Thus if the cylinder be turned the relative amounts of wire on the two cylinders can be varied at pleasure, and the wire is laid on them helically in a regular manner. The toothed wheel by which one of the cylinders is turned is connected with the axle by means of a spring previously wound up so as to give a couple tending to wind the wire on the cylinder. The wire is thus kept taut and the spires prevented from shifting on the cylinder. Rheostat.

The course of the current is along the wire on the paper covered cylinder, then to the bare cylinder. Thus the resistance in circuit is regulated by the amount of the wire on the former cylinder. A comparative estimate of this is given by the scale along which the nut moves.

The arrangement of screw and nut for guiding the wire above described seems to have been first used in a rheostat constructed by Mr. Jolin of Bristol.

Jacobi's Rheostat.

A simpler form of rheostat, first used by Jacobi, consists of a single cylinder of insulating material round which the wire is wound in a helical groove. A screw of the same pitch as the groove is cut in the axle of the cylinder, and works in a nut in one of the supports. The cylinder, when turned, moves parallel to itself, so that the wire is kept in contact with a fixed rubbing terminal. Another rubbing terminal rests on the axle, to which one end of the wire is attached.

Wheatstone's Bridge.

The method of comparing resistances of most general use is that known as Wheatstone's Bridge. The arrangement of conductors used is that shown in Fig. 34, with

Wheatstone's Bridge.

a battery, generally a single Daniell's or Menotti's cell, included in r_6, and a galvanometer in r_5. A much higher battery power is however sometimes required, especially in cable and other testing. The three conductors whose resistances are r_1, r_2, r_3 are coils of a resistance box provided with terminals so arranged that connections can be made at the proper places to form the bridge, for example as in Fig. 62, which shows a resistance box fitted up as a Wheatstone Bridge.

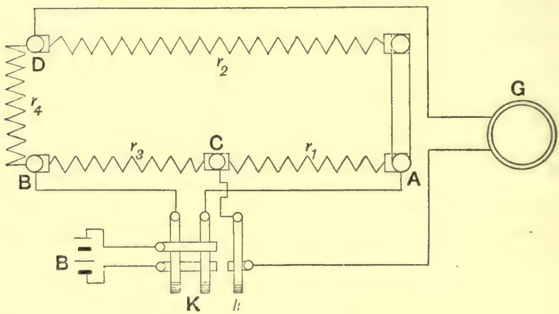

FIG. 70.

It will be easy to make out in Fig. 70 the terminals corresponding to A, B, C, D respectively of Fig. 34. Fig. 60 above shows a so-called "Post-Office Resistance Box" in which the battery and galvanometer keys are mounted on the cover, and permanently connected to the proper points inside the box; and Fig. 66 a Wheatstone Bridge arrangement of resistance slides.

The resistance to be compared is placed in the position BD, and convenient values of r_1 and r_2 are chosen,

while r_3 is varied until no current flows through the galvanometer. The value of r_4 is then found by (15) of Chapter II., which since γ_5 is zero, may be written Wheatstone's Bridge.

$$r_4 = \frac{r_2}{r_1} r_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If r_1 and r_2 are equal, r_4 is equal to r_3, and is read off at once from the resistance box.

In the practical use of Wheatstone's Bridge we have generally to employ a certain battery and a certain galvanometer for the measurement of a wide range of resistances; and it is possible if great accuracy is required so to choose the resistances of the bridge as to make the arrangement have maximum sensibility. An approximate determination is first made of the resistance to be measured. Call this r_4. It has been shown independently by Mr. Oliver Heaviside,* and by Mr. Thomas Gray,† that if the battery and galvanometer are invariable we should make Arrangement of Bridge for Greatest Sensibility. Practical Case—Invariable Battery and Galvanometer.

$$r_1 = \sqrt{r_5 r_6}, \quad r_3 = \sqrt{r_4 r_6 \frac{r_4 + r_5}{r_4 + r_6}}, \quad r_2 = \sqrt{r_4 r_5 \frac{r_4 + r_6}{r_4 + r_5}}.$$

If the resistances of the battery and galvanometer are at the disposal of the experimenter, then on the supposition that the resistance of the galvanometer may be taken equal to r_5, the most sensitive arrangement is that in which each of the resistances is equal to r_4.

* *Phil. Mag.* vol. xlv. (1873), p. 114.
† *Ibid.* vol. xii. (1881), p. 283.

Proof of Conditions of Maximum Sensibility.

To prove these results we may suppose that instead of (1) we have

$$c\frac{r_1}{r_3} = \frac{r_2}{r_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where c is a constant nearly equal to unity. Substituting the value of r_3, which (11) gives, in (25) of Chap. II., we get

$$\gamma_5 = \frac{E(1 - c)r_1r_4}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where D has the form given to it by this substitution and is therefore free from r_2. We have to find when this is a maximum, on the supposition that r_4, r_5, r_6, E, and c are constants, or which is the same but more convenient, when $E(1 - c)/\gamma_5$, (that is D/r_1r_4) is a minimum under the same conditions. Writing u for D/r_1r_4, and calculating the values of

$$\frac{du}{dr_1}, \ \frac{du}{dr_3}, \ \frac{d^2u}{dr_1^2}, \ \frac{d^2u}{dr_3^2}, \ \frac{d^2u}{dr_1dr_3},$$

we find, by equating the first two differential coefficients separately to zero, that either $r_1 = 0$, and $r_3 = 0$, or $r_1 = \sqrt{r_5r_6}$, and $r_3 = \sqrt{r_4r_6\frac{r_4 + r_5}{r_4 + r_6}}$. Substituting these values of r_1 and r_3 in the expressions for the three second differential coefficients, we find that the latter pair of corresponding values gives positive values to each of the expressions

$$\frac{d^2u}{dr_1^2}, \ \frac{d^2u}{dr_2^2}, \ \frac{d^2u}{dr_1^2}\frac{d^2u}{dr_2^2} - \left(\frac{d^2u}{dr_1dr_2}\right)^2,$$

which is the condition for a minimum, while the first pair of values gives neither a maximum nor a minimum.

Galvanometer Resistance Variable.

Let now the galvanometer resistance be capable of variation. We shall assume that the mass of wire in the galvanometer and the channel in which it is wound remain constant. When this is the case the electromagnetic force at the needle is, as will be proved in Vol. II., proportional to the square root of the resistance of the galvanometer. Hence for a given value of γ_5 the deflection of the needle may be put equal to $a\gamma_5\sqrt{r_5}$ where a is a constant. Hence

$$\delta = \frac{aE(1 - c)r_1r_4\sqrt{r_5}}{D} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

We have in this case to find when $D/r_1 r_4 \sqrt{r_5}$ is a minimum. This can be put into the form $(m + nr_5)/k\sqrt{r_5}$ where m, n, k, do not involve r_5. Equating the first differential coefficient of this quantity to zero, we get for a minimum $r_5 = m/n$, or after reduction $r_5 = r_1(r_4 + r_3)/(r_1 + r_2)$. This value of r_5 taken along with the already found values of r_1 and r_3 gives maximum sensibility to the bridge arrangement when the battery only is kept constant.

Galvanometer Resistance Variable.

Lastly, let the total area of the acting surfaces in the battery be given, while the resistance may be varied. In this case we have (p. 148 above) greatest sensibility when the resistance of the battery is made equal to the external resistance. If balance is nearly obtained, we may take as the external resistance between A and B the value $(r_1 + r_3)(r_3 + r_4)/(r_1 + r_2 + r_3 + r_4)$. If r_6 may be taken as the resistance of the battery alone (that is, if the electrodes joining the battery to A and B be made so massive that their resistance may be neglected) we have to arrange the battery so that

$$r_6 = \frac{(r_1 + r_3)(r_2 + r_4)}{r_1 + r_2 + r_3 + r_4}. \quad \ldots \quad (14)$$

Simultaneous with this we have the three equations already found, namely,

$$r_1 = \sqrt{r_5 r_6},\ r_3 = \sqrt{r_4 r_6(r_4 + r_5)/(r_4 + r_6)},\ r_5 = r_1(r_3 + r_4)/(r_1 + r_2).$$

From these it follows that

$$r_1 = r_2 = r_3 = r_4 = r_5 = r_6. \quad \ldots \quad (15)$$

It is to be carefully observed that for a given available electromotive force in the circuit not susceptible of alteration, the sensibility is greater the smaller r_6.

Unless in particular cases in which great accuracy is necessary, any convenient values of r_1, r_2 will give results sufficiently accurate for all practical purposes; but in arranging the bridge with these the following rule should be observed: of the resistances r_5, r_6 of the galvanometer and battery respectively, connect the greater so as to join the junction of the two greatest of

Practical Rule for Sensibility when Full Adjustment not needed.

Practical Rule for Sensibility when Full Adjustment not needed.

the four other resistances to the junction of the two least. This rule follows easily from (15) of Chap. II. For interchanging r_5 and r_6 we alter only the value of D, and calling the new value D' we get

$$D' - D = (r_5 - r_6)(r_1 - r_4)(r_3 - r_2) \quad . \quad . \quad (16)$$

The expression on the right will be negative if $r_6 > r_5$ and r_1, r_3 be the two greatest or the two least of the other resistances. Hence on this supposition the value of D has been diminished, and therefore the current through the galvanometer for any small value of $r_2 r_3 - r_1 r_4$ increased by making r_6 join the junction of r_1, r_3 to that of r_2, r_4. In cases in which the resistances in the bridge are large, a galvanometer of high resistance should also be used.

Arrangement of Keys.

In the practical use of the method the electrodes of the battery should be carried to the terminals of a reversing key, so that the testing current may be sent in opposite directions if desired through the resistances of the bridge. Also a single spring contact-key, which makes contact only when depressed, should be placed in r_5.

Mode of Operating.

These keys are convenient when arranged side by side, so that the operator placing a finger on each can depress one after the other. A convenient form of wire rocker with mercury cups, combining the two keys, may be easily made by the operator. When the bridge has been set up and a test is about to be made, the single key in r_5 is first depressed to test whether any deflection of the galvanometer needle is produced without closing the battery circuit. If there is a deflection, this must be due either to thermoelectric action in the galva-

nometer circuit, or to leakage from the battery to the galvanometer wires. The procedure in this case will be stated presently. If there is no deflection, the operator then opens the galvanometer circuit, depresses the key which completes the battery circuit, and immediately after, while the former key is kept down, depresses also the galvanometer key. After the circuits have been completed just long enough to enable the operator to see whether there is any deflection of the needle, the keys are released so as to break the contact in the reverse order to that in which they were made. This order of opening the circuits enables him to make a second observation of deflection without its being necessary to again send a current. It is easy to imagine and construct a form of contact-making key, which being depressed a certain distance completes the battery circuit, and on being depressed a little further completes the galvanometer circuit, and therefore on being released interrupts these circuits in the reverse order. This form of key is of use in the testing of resistance coils in which there is considerable self-induction. For general work, however, it is inconvenient, as the reverse order of making the contacts may have to be adopted. Again, in many practical operations, such as cable testing, &c., the contacts have to be made after different intervals of time in different cases.

Mode of Operating.

The object of thus completing and interrupting the battery circuit before that of the galvanometer is partly to avoid error from the effects of *self-induction*. When a current in a conducting wire is being increased or diminished, an electromotive force, the amount of which

Effect of Self-Induction.

Effect of Self-Induction.

depends on the arrangement of the conductor, is called into play, so as to oppose the increase or diminution of the current. The effect of this electromotive force is to produce, therefore, a weakening of the electromotive force of a battery for a very short time after the circuit is completed, and a strengthening during the very short interval in which the current falls from its actual value to zero at the interruption of the circuit. Its value is small when the wire is doubled on itself so that the two parts lie along side by side, the current flowing out in one and back in the other; but is very considerable if the wire is wound in a helix, and still greater if the helix contains an iron core. The electromotive force of self-induction is directly proportional to the rate of variation of the current in the circuit, and thus is explained the bright spark seen when the circuit of a powerful electro-magnet is *broken*.

Effect of Self-Induction in Wheatstone's Bridge.

If, then, one or more of the coils of a Wheatstone Bridge arrangement were wound so as to have self-induction, the electromotive force thus called into play would, if the galvanometer circuit were completed before that of the battery, produce a sudden deflection of the galvanometer needle when the battery circuit is closed. All properly constructed resistance coils are, as has been stated, made of wires which have been first doubled on themselves and then wound double on their bobbins, and have therefore no self-induction. The wire tested, however, and the connections of the bridge have generally more or less self-induction, the effect of which, unless the contacts were made as described above, might be mistaken for those of unbalanced resistance. This

mode of winding the coils also avoids direct electromagnetic effects on the coils on the galvanometer needle when the coils are placed near it.

Avoidance of Thermoelectric disturbance.

If on depressing the galvanometer key at first as described above a current is found to be produced by thermoelectric or leakage disturbance, and the spot of light is therefore displaced, the operator keeping down the galvanometer key depresses the battery key, and observes if there is any permanent deflection of the spot of light from its displaced position during the time that the battery key is kept down. This is easily distinguished from the sudden deflection due to self-induction, as that immediately dies away to zero as the current rises to its permanent value.

If the sudden deflection of the galvanometer, as it may be in the case of a dynamo or electro-magnet, is too violent and long continued, the reversing key of the battery should be used, the battery contact made first in each case, and the mean of the results taken.

Mode of operating continued.

When comparing a resistance the operator first observes the direction in which the mirror or needle is deflected when a value of r_3 obviously too great is used, and again when a much smaller value of r_3 is used. If the deflections are in opposite directions, the value of r_3, which would produce no deflection of the needle, lies between these two values, and the operator simply narrows the limits of r_3, until on depressing the galvanometer key no motion, or only a very small motion, of the needle is produced. It may happen, however, that the value of the resistance which is being compared may be between two resistances which have

Mode of operating continued.

the smallest difference which the box allows. Thus with a resistance box by which with equal values of r_1 and r_2 he cannot measure to less than $\frac{1}{10}$ of an ohm, he may either by making the ratio of r_1 to r_2, 10 to 1, or 100 to 1, obtain the values of r_4 to one or two places of decimals. Any inaccuracy in the relation of the arms of the bridge may be eliminated by reversing the arrangement, that is, interchanging r_1 and r_2, and r_3 and r_4, and taking the mean of the results.

Whatever be the ratio of r_1 to r_4, if he can read the deflections when first one and then the other value of

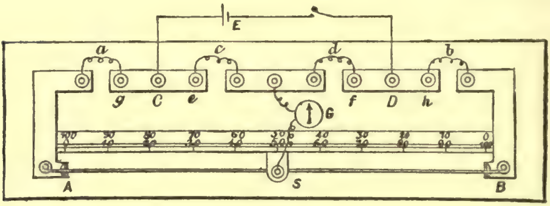

Fig. 71.

r_3 (between which r_4 lies, and which differ by only $\frac{1}{10}$ of an ohm) is used, he can find r_4 to another place of decimals by interpolation by proportional parts. For example, let the value 120·6 of r_3 produce a deflection of the spot of light of 6 divisions to the left, and 120·5 a deflection of 14 divisions to the right: the value of r_3 which would produce balance is equal to

$$120{\cdot}5 + {\cdot}1 \times 14 / (14 + 6) = 120{\cdot}57.$$

The most accurate form of Wheatstone's Bridge is

Kirchhoff's Bridge.

that introduced by Kirchhoff. In this an exact balance is obtained by moving a sliding contact-piece along a graduated wire which joins the two resistances r_1, r_4 of Fig. 70. A diagrammatic sketch of the arrangement is shown in Fig. 71. S is the sliding-piece, A, B the wire along which it slides. A, B is stretched in front of

FIG. 72.

scale a metre in length graduated to half-millimetres and doubly numbered, from left to right and from right to left. The coils a, c, d, b of the diagram have the respective resistances r_1, r_2, r_3, r_4. Fig. 72 shows a form of the instrument manufactured by Messrs Elliott Bros.

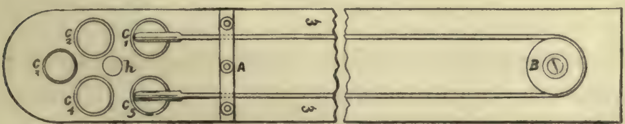

FIG. 73.

T. Gray's Bridge.

Fig. 73 shows an easily-made and cheap form of wire bridge devised by Mr. T. Gray. w, w is the wire, made of platinoid or German silver, which is stretched above, but not in contact with, a base-board, passes round the insulating and supporting vulcanite block B from the

T. Gray's Bridge.

mercury cup c_1 to the other c_5. A vulcanite crossbar A clamps the wire in position near the cups. If the wire be long several such crossbars may be used. Each end of the wire is soldered to a large mass of copper, bent as shown in Fig. 74 so as to dip into a mercury cup without any risk of contact of the mercury with the soldered junction. The cups should be of copper, and may conveniently be made of the form shown in the figure, and fixed in holes in the wooden or ebonite supporting-block. The ends of the copper pieces

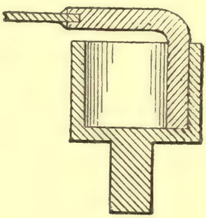

FIG. 74.

dipping into them should be carefully squared and bear against the copper bottoms.

The wire is divided into parts of equal resistance by a process of calibration (p. 342 *et seq.*, below) and marks indicating these parts are made on a rule attached to the base-board, along which the contact-piece slides. A movable scale is used to subdivide the space between two divisions.

On a plate of ebonite or well-paraffined hard wood are fixed mercury cups c_2, c_3, c_4, made as just described.

The auxiliary resistances r_1, r_4 of the bridge when required are placed between c_1 and c_2, c_5 and c_4, while the wires to be compared connect c_2 and c_3, c_3 and c_4. Since the wire w, w can be made long, the auxiliary resistances are not frequently required. When they can be dispensed with, c_2 and c_4 are removed and c_3 placed in the socket h, and the wires to be compared are then placed between c_3 and c_1, c_3 and c_5.

T. Gray's Bridge.

This method of obtaining balance was used by Matthiessen and Hockin in the very careful comparisons of resistance made by them in their work as members of the British Association Committee on Electrical Standards; and it was found by these experimenters that an alloy of 85 parts of platinum with 15 parts of iridium formed an excellent material for the graduated wire. This alloy, they found, did not readily become oxidized. Platinum-silver alloy is however frequently employed.

The contact piece is generally a well-rounded edge of steel with a slight notch to receive the wire. The knob pressed by the operator bends a spring which presses the contact piece with just sufficient pressure against the wire. A turning bar can be put into position to keep down the contact when desired. The sliding piece carries a vernier which enables fractions of a division to be read on the scale.

The method of testing by this instrument is precisely the same as by the ordinary Wheatstone Bridge, except that when balance has been nearly obtained in the usual way, by varying the relation of the resistances r_1, r_2, r_3, for a particular position of the sliding piece, an exact balance is obtained by shifting the sliding piece in the

Calibration of a Wire: Method of Matthiessen and Hockin.

proper direction along the wire. Supposing that the resistance of the wire per unit of length has been determined for different parts of the wire, and that the resistances of contacts have been determined (p. 357) and allowed for, the value of r_4 is at once found by taking into account the resistances of the segments of the wire AB on the two sides of the point, contact at which gives zero deflection.

Calibration of a Wire: Method of Matthiessen and Hockin.

The wire AB may be "calibrated" by one of the following methods. The first is that which was employed by Matthiessen and Hockin.* Let r_1 and r_4 (a, b in Fig. 71) be such resistances that balance is obtained at some point P in AB, and let r_2, r_3 (c, d in Fig. 71) be two coils, differing in resistance by say $\frac{1}{10}$ per cent. Let $r_1 + a$ be the total resistance, including contacts between C and P, and $r_4 + \beta$ that between D and P. Now alter r_1 by inserting a short piece of wire. This will shift the zero point along the wire through a certain distance to the left. Balance so as to find this point, which call P_1; then interchange r_2 and r_3, and balance again, and call the second point thus found P_2. Let z denote the resistance between P and P_1, z' the resistance between P and P_2, x the resistance of the short piece of wire added to r_1, and l the length of wire between P_1 and P_2. We have plainly the two equations,

$$\left.\begin{aligned}\frac{r_1 + a + x - z}{r_2} &= \frac{r_4 + \beta + z}{r_3}\\ \frac{r_1 + a + x - z'}{r_3} &= \frac{r_4 + \beta + z'}{r_2}\end{aligned}\right\} \quad . \; . \; . \quad (17)$$

* *Reports on Electrical Standards*, p. 119.

from which we obtain for the resistance per unit of length between P_1 and P_2, Mathiessen and Hockin's Method.

$$\frac{z - z'}{l} = \frac{r_2 - r_3}{l(r_2 + r_3)}(r_1 + r_4 + \alpha + \beta + x) \quad . \ (18)$$

The value of x is easily obtained with sufficient accuracy from either of equations (17), as z is approximately known from the known resistance of the whole wire. In this way the resistance per unit of length at different parts of the wire can be easily found, and, if necessary, a table of corrections formed for the different divisions of the scale.

Professor Carey Foster has given the following method for the calibration of the bridge wire. The arrangement is shown diagrammatically in Fig. 75. The battery shown in Fig. 71 is removed, and two equal copper bars are attached at C, D (Fig. 71), at right angles to the bars of the bridge at those points. Between the extremities of these is stretched a second slide wire. Or the slide wire of a second bridge, from which all other connections have been removed, may be connected to C and D by wires from the end bars to which it is attached. In place of the coils c, d of Fig. 71, and the middle bar of the bridge is substituted a single Daniell's or other cell. One terminal of the galvanometer is connected to a sliding piece on the wire W, the other to a sliding piece on the other wire, W'. In place of r_1 and r_4 are substituted two small resistances, one simply a piece of thick wire c, the other a resistance g, equal to that of a convenient portion say from 80 to 100 millimetres of the bridge wire. The former of these has Carey Foster's Method.

Carey Foster's Method.

been called the connector, the latter the gauge. They are connected to the bridge by mercury cups in the manner described on p. 340 above, and some form of switchboard is usually employed to effect the interchanges described below.

Supposing the gauge placed first on the left and the connector on the right, the slide on W is moved close up to the extremity B, and balance is obtained by placing the slider on W' at some point near D. The gauge and connector are then interchanged, and balance is again obtained by shifting the slider on W' towards the left to some point b.

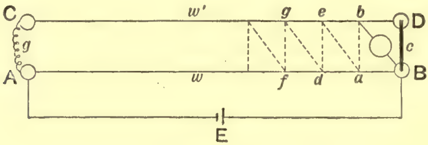

FIG. 75.

The gauge and connector are again interchanged, and balance obtained by shifting the slide on W to the left, and so on until both wires have been traversed almost completely from end to end. The distance through which the slider is moved at each interchange of the resistance is read off, and gives, as we shall now show, a determination of the average resistance per unit of length over that portion of the wire. Let P and P' be points of contact on W and W' when balance is obtained, let the permanent resistances included with W, W' at the left-hand ends be denoted by a, b, and at the

right ends by a', b' respectively, the resistance of the connector by c, of the gauge by g, of the wire from A to P by z, of the whole wire by w, of the wire W' from A' to P' by z', and of the whole wire by w'. If the connector be on the left and the gauge on the right, we have by

Carey Foster's Method.

$$\frac{c + a + z}{a' + z'} = \frac{g + b + w - z}{b' + w - z'} \quad . \quad . \quad . \quad (19)$$

and if the gauge and connector be interchanged so that r receives a new value r_1,

$$\frac{g + a + z_1}{a' + z'} = \frac{c + b + w - z_1}{b' + w - z'} \quad . \quad . \quad . \quad (20)$$

From these equations we get at once

$$g - c = z_1 - z \quad . \quad . \quad . \quad . \quad . \quad (21)$$

that is, the steps along W have each a total resistance equal to $g - c$, a result evident without calculation at all.

Again, supposing the gauge at first on the left, and next on the right, the slider on W' is shifted, and we get the equations

$$\frac{a' + r'}{g + a + r} = \frac{b' + w' - z'}{b + c + w - z}$$

$$\frac{a' + r'}{c + a + r} = \frac{b' + w' - z'_1}{b + g + w - z}.$$

These give

$$r_1 - r'_1 = (g - c)\,\frac{a' + b' + w'}{a + b + c + g + w'}. \quad . \quad . \quad (22)$$

Carey Foster's Method.

The quantities on the right-hand side are all constants, and therefore the wire W' is thus divided into parts of equal resistance. From the known resistance of the whole wire, which can be found as shown on p. 354 below, the resistance of each part can be found. The steps on each wire are thus steps of equal resistance.

The following are the actual results obtained in the calibration of the slide-wire of a bridges performed in the Physical Laboratory of the University College of North Wales by the method just described.

Parts of the wire of equal resistance ($= r$).		Resistances of the parts included between the corresponding readings.	
Readings (taken zero at right hand end).	Lengths l.	Readings.	Resistance $= \frac{10r}{l}$.
0...10·59	10·59	0... 10	·94429 r
9·79...20·35	10·56	10... 20	·94697 ,,
19·70...30·26	10·56	20... 30	·94697 ,,
29·84...40·41	10·57	30... 40	·94607 ,,
39·69...50·22	10·53	40... 50	·94967 ,,
49·71...60·27	10·56	50... 60	·94697 ,,
59·80...70·35	10·55	60... 70	·94787 ,,
69·82...80·32	10·50	70... 80	·95238 ,,
79·86...90·38	10·52	80... 90	·95057 ,,
89·41...99·97	10·56	90...100	·94697 ,,
		0...100	9·47873 r

The numbers in the right-hand column are taken from tables. The results are of course not correct to the number of decimals given.

It will be noticed that the second reading in any line of the first column is not exactly the same as the first reading in the next line. This was caused through its being difficult to balance by adjusting the contact on the auxiliary wire. Balance was therefore obtained after a step was taken along the auxiliary wire by moving the slider through a short distance on the wire which was being calibrated. Carey Fosters' Method.

The value of r found as described below, p. 355 was ·0452 ohm. From this the resistance of the part of the wire between two readings of the scale is found as shown in the table.

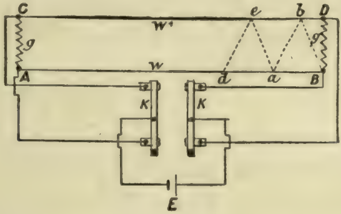

FIG. 76.

A modification of this method which works well in practice and avoids some difficulties has been made by Mr. Thomas Gray. The two wires W, W', are arranged parallel to one another as in Fig. 76, and are connected at the ends A, C and B, D by two equal small resistances of suitable amount g. The equality of these resistances can be tested with great ease and delicacy by connecting the battery at A, B, and balancing with the galvanometer between a point on W and another T. Gray's First Method by Bridge.

T. Gray's First Method by Bridge.

on W', then transferring the battery contacts to C, D and observing if the balance is disturbed. If it is not the resistances are equal. When the resistances have been adjusted to equality, the battery is brought into contact at A and D and balance is obtained by placing one galvanometer terminal close to B on W, and the other at b on W'. The battery contacts are then transferred to B and C, and balance is obtained by shifting the terminal of the galvanometer on W to some point a while that on W' is kept at b. The battery contact is then transferred to A, D, and balance obtained by moving the terminal on W' so that the points of contact are a, d, and so on.

The readings on the graduated scales are taken for the successive points of contact, and divide each wire, as will be shown presently, into steps each of resistance g.

The contact of the battery at A, D or B, C can be made by means of two simple rockers K, K, working between mercury cups or ordinary metal contacts, or by means of any simple key. This renders unnecessary any mercury cup switchboard arrangement for transferring coils.

Thus the method has the great advantage that the contacts are all permanent except those of the battery and the sliders, no one of which of course introduces any error.

Let contact be made by the battery at A and D, and balance be obtained with the galvanometer at points a and e on the wires W and W', then calling as before z, z' the resistances of the wires between A and a, C and e respectively, and w, w' the resistances of the whole

wires, we have, neglecting (which will not affect the result) constant resistances of connecting bars, &c.,

T. Gray's First Method by Bridge.

$$\frac{w - z + g}{z} = \frac{w' - z'}{z' + g} \quad . \quad . \quad . \quad . \quad (23)$$

Let the battery now be transferred to B and C and balance be obtained at d and e. Denoting the resistance between A and d by z, z, we again have

$$\frac{w - z_1}{z_1 + g} = \frac{w' - z' + g}{r'} \quad . \quad . \quad . \quad . \quad (24)$$

Equations (23) and (24) give

$$z - z_1 = 2g\frac{w + g}{w' + g} \quad . \quad . \quad . \quad . \quad (25)$$

or the steps along W are steps of equal resistance. The same can of course be proved for W'.

T. Gray's Second Method by Bridge.

Fig. 77 shows another arrangement devised by Mr. T. Gray, which has also the advantage of having permanent gauge contacts. A and C are joined by a short thick wire or connector c, while B and D are

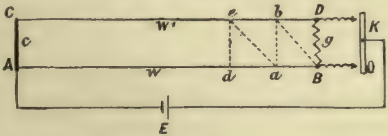

Fig. 77.

joined by a gauge g. The battery contact is made alternately at B and D by a key K, and balance is obtained by moving the galvanometer contact on W and W' alternately. This arrangement is rather simpler to set up than the former but gives steps the

T. Gray's Second Method by Bridge.

resistances of which are in geometrical progression. These though perhaps not quite so convenient as steps of equal resistance give the required calibration with sufficient exactness; for if the increase in length of the steps which takes place from one end to the other prove inconvenient, the calibration may be repeated in the reverse direction. According as the calibration of W is to be begun at B or at A, and according as that of W' is to be begun at C or at D, the first contact of the battery must be made at D or at B.

Let a and e, e and d, d and f, f and g, be four successive positions of the galvanometer terminals for which balance is obtained, and let z_1, z_2, z_3 be the resistances on W from A to a, A to d, A to g respectively, z'_1, z'_2, the resistances on W' from C to e, C to f respectively. Then if the battery be connected to B for the contacts a, e, we get for these successive positions the equations

$$\frac{w - z_1 + g}{z_1} = \frac{w' - z'_1}{c + z'_1}, \quad \frac{w - z_2}{z_2} = \frac{w' - z'_1 + g}{c + z'_1},$$

$$\frac{w - z_2 + g}{z_2} = \frac{w' - z'_2}{c + z'_2}, \quad \frac{w - z_3}{z_3} = \frac{w' - z'_2 + g}{c + z'_2}.$$

These give

$$\frac{z_1 - z_2}{z_2 - z_3} = \frac{(w + g)\,(w' + g + c)}{w(w' + c)} \quad . \quad . \quad (26)$$

a constant ratio. An equation similar to (27) could of course be found for two successive steps on W'.

T. Gray's Method by Differential Galvanometer.

Mr. T. Gray has also suggested the following very simple method which is practically identical with that previously used by Prof. Tait for the comparison of low resistances (see below, p. 371). It has been found to give

excellent results. The arrangement is shown in Fig. 78. The wire W to be calibrated is joined up in series with a single Daniell or storage-cell, with resistance * just enough to prevent an excessive current from flowing in the circuit, if the cell is of very low resistance. It is advantageous to use a differential galvanometer. A pair of electrodes (not shown in the figure) from one coil of the galvanometer are attached to terminals which are fixed for the time in contact with the wire W at two points close to one end, or at the two ends of a convenient gauge wire in the same circuit. The resulting deflection is annulled by placing electrodes from the

T. Gray's Method by Differential Galvanometer.

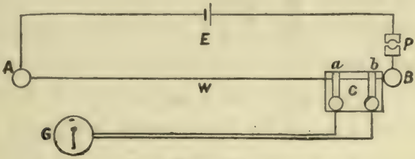

FIG. 78.

other coil in contact at other two points on the wire. For the latter pair of terminals it is convenient to have a sliding piece with two contacts, one for each terminal, with index marks opposite the scale at a distance apart nearly equal to that between the contacts. A part, carrying one index mark, and the corresponding contact making point, is made movable with a fine screw, so that the distance between the contact pieces may be increased or diminished by a small amount to enable

* Since this resistance need not be of known amount it may be that of a convenient portion of a rheostat wire included in the circuit (see p. 328 above).

T. Gray's Method by Differential Galvanometer.

balance to be obtained. This double contact piece is simply moved along by steps each equal to the distance existing between the contacts, which insures that the nearer point of contact with the wire for the new position is coincident with the farther point of contact for the last position, the words nearer and farther being used relatively to the end of the wire at which the calibration is begun. The farther contact is the movable one, and this is adjusted for compensation at each step. Plainly the wire is thus divided into steps of equal resistance. The contacts may also be made by fine wires, caused by light weights to press on the wire which is being calibrated. For a simple test of the uniformity of a wire, the contacts may be simply slided along the wire, and the change in deflection noted. The distance through which the adjustable electrode would have to be moved in order to reduce the deflection to zero for any step, may be inferred from the deflection produced by transferring the movable terminal once for all by the micrometer screw through a measured distance along the wire.

By thus displacing the movable contact piece through a small distance from the position for balance, and observing the deflection, the sensibility of the method can be ascertained and increased or diminished by altering the resistance or the electromotive force in circuit.

The theory of this method is obvious. The difference of potentials between the ends of the movable electrodes when balance is obtained is always in the same ratio to the difference of potentials between the fixed electrodes

however the electromotive force of the battery may vary; hence the distance between them for balance is independent of the current flowing through the battery.

Modification to suit Ordinary Galvanometer.

The same method can be used with an ordinary galvanometer by bringing the spot of light back to the zero position by means of a controlling magnet. The electrodes are then shifted step by step, and any change in the deflection is shown by the deflection of the spot of light from zero.

Precaution against Thermoelectric Disturbance.

In all these methods disturbance from thermoelectric currents, due to accidental differences of temperature at the surfaces of contact of dissimilar metals, is to be avoided by using the reversing key in the battery circuit, and balancing for both directions of the current; and if there is any difference of position for balance, taking the mean position as the correct one.

Comparison of Coils by Slide-wire Bridge.

The slide-wire bridge may be used for the accurate comparison of resistance coils with a standard, say for the adjustment of single ohms with a standard ohm. Fig. 71 (p. 338 above) shows the arrangement adopted. r_1 and r_4 are the resistances of the coils a, b to be compared, and are nearly equal. r_2 and r_3 are the resistances of the two coils c, d, and are each nearly equal to r_1 or r_4. The connections are made by mercury cups as already described. Balance is obtained with the contact-piece somewhere near the middle of the slide-wire. The coils r_1, r_4, are then interchanged and balance again obtained. By (21) above we have

$$r_1 - r_4 = z_1 - z_2 \quad . \quad . \quad . \quad . \quad . \quad (27)$$

where z_1, z_2 are the resistances of the wire from A to

Comparison of Coils by Slide-wire Bridge.

the point of contact in the two cases. If ρ be the resistance per unit of length for the whole wire, s_1, s_2 the distances (reduced, if necessary, by calibration, as shown above, to distances along a wire of uniform resistance ρ per unit of length) measured along the wire from A, we have

$$r_1 - r_4 = \rho(s_1 - s_2) \quad . \quad . \quad . \quad . \quad (28)$$

These results are evidently free from any uncertainty as to the resistance of the junctions of the slide-wire to the copper bars at its ends, and from any error due to want of correspondence between the index mark on the sliding-piece and the point of contact.*

If a separate experiment be made with a coil of accurately known resistance r_1, just a very little less than that of the whole wire, and a second conductor of resistance r_4 so small that it may be neglected, the value of ρ may be obtained from the equation

$$\rho = \frac{r_1}{s_1 - s_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

If the coils compared are too unequal to allow balance to be made on the wire, a series of intermediary coils may be obtained, so as to give a gradual descent from one coil to the other.

Method of finding Resistance of Slide Wire.

The resistance of the wire between any two readings may also be determined by the following method, which is

* The resistance of a coil may be accurately adjusted to any required value by first making it slightly too great, and then joining it in multiple arc with a thin wire cut so as to give as nearly as possible the required correction. If the observed resistance be r_4, and that required r_1, the resistance of the correcting wire is $r_1 r_4/(r_4 - r_1)$.

Method of finding Resistance of Slide Wire.

due to Mr. D. M. Lewis. The total resistance of the wire is approximately found by measuring it with an ordinary bridge consisting of a post-office set of coils or other available form of resistance box. Two coils are then made, the resistance of each of which is less than unity by a quantity which is nearly equal to, but not greater than, the total resistance of the wire. These can be also made by means of an ordinary resistance box. Let R_1, R_2 be the as yet not accurately known resistances of these coils. Each is tested as follows in the slide wire bridge against a unit coil, a standard ohm for example. The unit coil is first placed in the position a of Fig. 71, and one of the two resistances, R_1 say, is placed in the position b. The connections should be made by mercury cups as already described. In the positions marked c, d are placed permanently two coils of nearly equal resistance. The magnitudes of these need not be known, but should not be greater than one or two units. Balance is obtained with the slide S at a point near the end B of the slide wire, and the reading on the slide scale is taken. The coil R_2 and the unit are then interchanged, and balance obtained with the slide near A. The difference of the two readings gives the length of wire intercepted between them, and this must be equal in resistance to $1 - R_1$.

The other coil R_2 is now substituted for R_1 and two readings for which balance is obtained taken in the same way. These give a length of the wire the resistance of which is $1 - R_2$.

The two resistances are now put together in series and tested against the unit in precisely the same way,

Method of finding Resistance of Slide Wire.

and give between the two readings taken a length of wire of resistance $R_1 + R_2 - 1$.

Now from a previously made calibration of the wire the resistances of the three portions of the wire thus observed can be obtained in terms of the resistance of the calibration-step, and three equations are thus available for the determination of the three unknown quantities R_1, R_2, and r, the resistance of the step used in calibration, as in p. 346 above. The following table gives the results of this process applied to the slide-wire the calibration of which is given above.

Positions of the Resistances.		Readings on Slide Wire.	Resistances between these readings in terms of r. Obtained from Table, p. 346 above.
Left.	Right.		
R_1 1	1 R_1	1·40 97·72	$9{\cdot}13063r$ [9·47875 − ·13220 − ·21590 = 9·131]
R_2 1	1 R_2	0·14 98·97	$9{\cdot}36797r$ [9·47873 − ·01322 − ·09754 = 9·368]
$R_1 + R_2$ 1	1 $R_1 + R_2$	69·70 31·45	$3{\cdot}62498r$ [3·79058 − ·02843 − ·13717 = 3·625]

Here
$$1 - R_1 = 9{\cdot}131r,\ 1 - R_2 = 9{\cdot}368r$$
$$R_1 + R_2 - 1 = 3{\cdot}625r$$

and therefore

$$r = \frac{1 - R_1}{9{\cdot}131} = \frac{1 - R_2}{9{\cdot}368} = \frac{R_1 + R_2 - 1}{3{\cdot}625} = \frac{1}{22{\cdot}124} = {\cdot}0452.$$

Substituting this value of r in the first two equations we find R_1 and R_2. This can be used, as shown at p. 346 above, to find the resistance of the portion of the wire between any two readings of the scale.

An accurate comparison of two nearly equal resistances, for example a unit with its copy, can be obtained by making the r_2 and r_3 to be compared occupy the positions c, d, of Fig. 71. Balance is first obtained with r_2 and r_3 in one pair of positions, then they are interchanged and balance again obtained. Assuming that the permanent resistances are included in r_1, r_4, r_2, r_3, and giving z_1, z_2 the same meanings as at p. 345 above, we have

Comparison of two Nearly Equal Resistances.

$$\frac{r_2}{r_3} = \frac{r_1 + z_1}{r_4 + w - z_1} = \frac{r_4 + w - z_2}{r_1 + z_2}$$

$$= \frac{r_1 + r_4 + w + z_1 - z_2}{r_1 + r_4 + w - (z_1 - z_2)}$$

and therefore

$$\frac{r_2 - r_3}{r_3} = \frac{2(z_1 - z_2)}{r_1 + r_4 + w - (z_1 - z_2)} \quad . \quad . \quad (30)$$

Hence the greater $r_1 + r_4$ the greater $z_1 - z_2$. Thus by choosing a pair of resistances as nearly equal as possible, and sufficiently great, r_2 and r_3 may be compared to any needful degree of accuracy.

The permanent resistances, α, β say, corresponding to the coils a, b of Fig. 71, may be estimated by the following method, by which two low resistances can be measured when the ratio of two others is accurately known. Let the resistances r_2, r_3 of c, d in Fig. 70 have the known ratio μ. We shall suppose r_1 and r_4 to be so low

Measurement of two Low Resistances.

Measurement of two Low Resistances.

resistances that, with a value of μ differing considerably from unity, balance can be found on the wire. Balance is obtained with the coils in the positions c, d, shown in Fig. 71; then r_2 and r_3 are interchanged, and balance is again obtained. We have

$$\mu = \frac{r_1 + z_1}{r_4 + w - z_1} = \frac{r_4 + w - z_2}{r_1 + z_2}.$$

From these equations we obtain

$$r_1 = \frac{z_1 - \mu z_2}{\mu - 1}, \quad r_4 = - w + \frac{\mu z_1 - z_2}{\mu - 1} \quad . . . (31)$$

If thick copper pieces be substituted for the coils a, b of Fig. 71, their resistances, if the connections as is understood are made with proper mercury cups, may be taken as zero, and α and β are approximately given by (31). The values of α, β thus obtained may be used for the correction of the values of r_1, r_4 found as just described. This correction will not be appreciably affected by the unknown permanent resistances corresponding to the coils c, d, if r_2, r_3 are taken moderately large so that the actual ratio may be taken as equal to their known ratio.

Comparison of very low Resistances.

Neither of the arrangements of Wheatstone's Bridge described above is at all suitable for the comparison of the resistances of short pieces of thick wire or rod, for example, specimens of the main conductors of a low resistance electric light installation, the resistances of which are so small as to be comparable with, if not less than, the resistances of the contacts of the different wires by which they are joined for measurement. To obtain an accurate result in such a case, we must

compare, directly or indirectly, the difference of potentials between two cross-sections in the rod which is being tested, with the difference of potentials between two cross-sections in a standard rod, while the same current flows in both rods, in a direction parallel to the axis at and everywhere between each pair of cross-sections.

Thomson's Bridge with Secondary Conductors.

Sir William Thomson has so modified Wheatstone's Bridge, by adding to it what he has called *secondary conductors*, as to enable it to be used, with all the convenience of the ordinary arrangement, for the accurate comparison of the resistance of a foot or two of thick

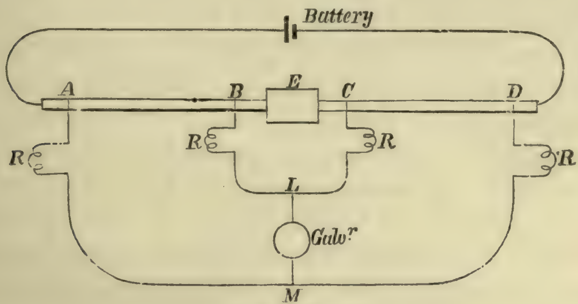

FIG. 79.

copper conductor with that between two cross-sections in a standard rod. The arrangement is shown in Fig. 79. *CD* are two cross-sections, at a little distance from the ends of the conductor to be tested, and *AB* are two similar cross-sections of the standard conductor. These rods are connected by a thick piece of metal, so that the resistance between *B* and *C* is very small, and the terminals of a battery of low resistance are applied

Thomson's Bridge with Secondary Conductors.

at the other extremities of the rods as shown. The sections BC are connected also by a wire BLC, and the sections AD by a wire AMD, in each case by as good metallic contacts as possible. BLC and AMD may very conveniently be wires, along which sliding contact-pieces L and M can be moved, with resistances R, R, R, R of half an ohm or an ohm each, inserted as shown in the figure. The sections A, D are so far from the ends of the rods, and the wires AMD, BLC are made of so great resistance (one or two ohms is enough in most cases), that the current throughout the portions of the conductors compared is parallel to the axis, and the effect of any small resistance of contact there may be at A, B, C, D is simply to increase the effective resistance of BLC and AMD by a small fraction of the actual resistance of the wire in each case. The terminals of the galvanometer G are applied at L and M, and the circuits of the galvanometer and battery are completed through a double key as in the ordinary bridge. A reversing key is inserted in the battery circuit as in other cases, to enable the comparison to be made with both directions of current.

Let the resistances AM, DM be denoted by r_1, r_3; BL, CL by a, b, AB, CD by r_2, r_4, and BC by s. Suppose r_1 and r_3 to be varied by moving the sliding piece at M till no current flows through the galvanometer. To find the relation which must hold among the resistances when this is the case, we may suppose the point L connected by a bar of zero resistance, with the cross-section of E, which is at the same potential as L. Call this cross-section K. The resistance of the portion of BC to

the left of K is $as/(a+b)$, and of the portion to the right $bs/(a+b)$. The resistance between B and KL is therefore $\{a^2s/(a+b)\}/\{a+as/(a+b)\}$, or $as/(a+b+s)$, and similarly that between C and KL is $bs/(a+b+s)$. Hence by (1) we have

Thomson's Bridge with Secondary Conductors.

$$r_3\left(r_2+\frac{as}{a+b+s}\right)=r_1\left(r_4+\frac{bs}{a+b+s}\right),$$

or

$$r_1r_4-r_3r_2=\frac{s}{a+b+s}(ar_3-br_1)\quad . \quad . \quad . \quad (32)$$

Now s has been supposed very small in comparison with $a+b$, and a and b can be easily chosen so as to make ar_3-br_1 approximately equal to zero. Hence equation (4) reduces to

$$r_1=\frac{r_3}{r_4}r_2\quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

the formula found above for the ordinary Wheatstone Bridge.

The apparatus illustrated in Fig. 80 is convenient for the carrying out of this method in practice. On a massive sole plate of iron, P, are mounted two vertical guide-rods of copper, A, A, and parallel to these the rods to be compared, viz., a standard rod C, and the rod to be tested C_1. C, C_1 are supported with their lower ends in two mercury cups cut in a single block of copper. This block corresponds to the piece E in Fig. 79. The upper ends of C, C_1 are fixed in screw blocks of copper, t, t, to which also are attached the terminals of a constant battery B of low resistance. A reversing key K is interposed between t, t and the

Practical Apparatus for Thomson's Bridge.

Practical Apparatus for Thomson's Bridge.

battery. A scale D graduated along its two edges nearly fills the space between the rods C, C_1.

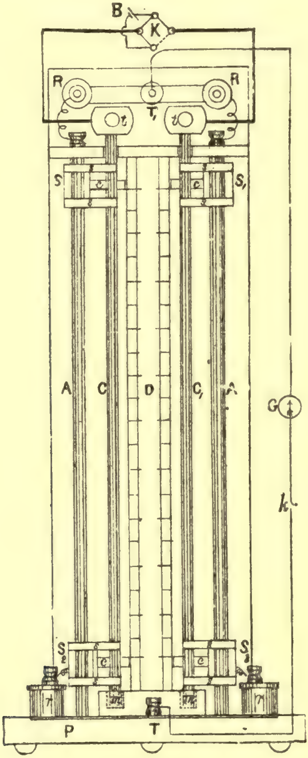

FIG. 80.

A pair of resistance coils, r, r, are fixed to the sole plate, and have one terminal of each connected by a strip of copper, which also carries the terminal screw T.

The other terminals of these coils are fixed to two copper sliders, S_2, S_3, which move along, but are insulated from, the guide-rods, and carry contact pieces c, c, each of which is bevelled off to a knife-edge on a level with its upper side. This knife-edge is pressed against the corresponding rod by springs s, s, which are insulated so as not to touch the rods. The coils r, r are attached directly to the contact pieces c, c. Thus $S_2\ r\ T\ r\ S_3$, corresponds to the partial circuit $BRLRC$ of Fig. 79.

Practical Apparatus for Thomson's Bridge.

Near the upper ends of C, C_1 is a similar arrangement of sliders, S, S_1 with spring contacts and attached coils, R, R. These coils are connected by a copper strip which carries the terminal T_1. The coils R, R are attached to the upper ends of the guide-rods A, A, and through these to the sliders S, S_1. The guide-rods are so thick that no appreciable change is made in the ratio of the resistances of the parts of the partial circuit SRT_1RS_1 on the two sides of T_1 by varying the positions of the sliders. This partial circuit corresponds to $ARMRD$ of Fig. 79.

Each pair of coils, r, r and R, R, may be wound on a single bobbin with advantage. The arrangement is thereby rendered more compact, and there is less risk of error from difference of temperature between the bobbins, or of thermoelectric disturbance between their terminals.

Between T and T_1 is placed the galvanometer G, which is provided with a simple key k, placed for convenience in the actual arrangement beside the reversing key K.

Practical Apparatus for Thomson's Bridge.

In the use of the instrument the rods to be compared are placed in position, and the sliders on the rod of lower resistance are placed so that their upper edges, and therefore their knife-edges, are opposite the lowest and uppermost divisions of the scale. The lower contact piece on the other rod is placed with its upper edge opposite the lowest division of the scale on that side. The upper contact piece on the same rod is then shifted until no current flows through the galvanometer. Balance is obtained for both directions of the current, and the mean position of the slider taken, to eliminate error from thermoelectric disturbance.

A number of standard rods of different thicknesses are provided with the instrument in order that nearly equal ratios may be obtained over a wide range of low resistances.

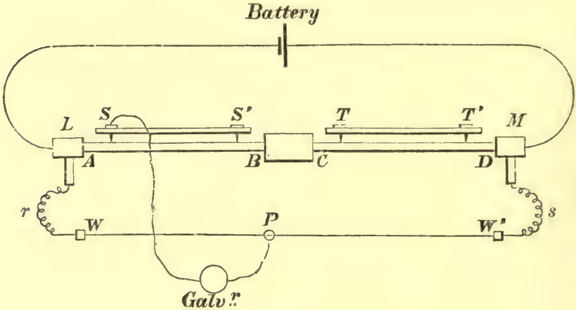

Fig. 81.

The following method was used for the same purpose by Messrs. Matthiessen and Hockin in their researches on alloys. *AB, CD,* Fig. 81, are the two rods to be

compared. They are connected in circuit with two coils of resistances r, s, which have between them a graduated wire WW, as in Kirchhoff's bridge. SS' are two sharp knife-edges, the distance of which apart can be accurately measured, fixed in a piece of dry hard wood or vulcanite, and connected with mercury cups on its upper side. This arrangement is placed on the conductor AB, so that the knife-edges making contact include between them a length SS' of the rod. TT' is a precisely similar arrangement placed on CD. One terminal of the galvanometer is applied at S, and the resistances r, s adjusted so that a point P on the wire which gives balance is found for the other terminal. The terminal of the galvanometer is shifted to S', and a second point P' found by varying the resistances of the coils from r_1, s_1 to r'_1, s'_1 in such a manner as to keep the sum $r+s$ constant. Similarly balance is found for TT' with values r_2, s_2, r'_2, s'_2, for the resistances of the coils, fulfilling the condition that the sum $r+s$ is the same as in the former case. Let a, b, c, d, k denote the resistances between L and S, L and S', L and T, L and T', L and M respectively; $\alpha, \beta, \gamma, \delta$ the resistance between W and P in the four cases, κ the resistance of the whole wire WW'. We have by (1)

Matthiessen and Hockin's Method.

$$\frac{a}{k-a} = \frac{r_1+\alpha}{s_1+\kappa-\alpha},$$

and therefore

$$\frac{a}{k} = \frac{r_1+\alpha}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

Matthiessen and Hockin's Method.

Where

$$R = r + s + \kappa,$$

similarly

$$\frac{b}{k} = \frac{r'_1 + \beta}{R}.$$

Therefore

$$\frac{b - a}{k} = \frac{r'_1 - r_1 + \beta - a}{R} \quad . \quad . \quad . \quad . \quad (35)$$

In the same way we get

$$\frac{d - c}{k} = \frac{r'_2 - r_2 + \delta - \gamma}{R} \quad . \quad . \quad . \quad . \quad (36)$$

and combining the last two equations we get for the ratio of the resistances of the conductors between the pairs of knife-edges—

$$\frac{b - a}{d - c} = \frac{r'_1 - r_1 + \beta - a}{r'_2 - r_2 + \delta - \gamma} \quad . \quad . \quad . \quad (37)$$

Method of Direct Comparison of Potentials.

The following method of comparing resistances is in principle the same as Thomson's Bridge with secondary conductors, and Matthiessen and Hockin's method described above, as, like them, it consists in comparing the difference of potentials between two cross-sections near the ends of the conductor to be tested with the difference of potentials between two cross-sections in a standard conductor, when the same uniform current is flowing in both. It is, however, more readily applicable in practice, and is very useful for a great many purposes, as for example, in the testing of the armatures or magnet coils of machines, in the estimation of the resistances of contacts, and in the deter-

Method of Direct Comparison of Potentials.

mination of the specific conductivities of thick copper wires or rods. All that is required is a small battery, a suitable galvanometer of sufficient sensibility, and two or three resistance coils of from $\frac{1}{5}$ ohm to 1 ohm. These coils may very conveniently for many purposes be made of galvanised or tinned iron wire of No. 14 or 16 B.W.G., wound round a piece of wood $\frac{1}{2}$ inch thick, from 8 to 10 inches broad, and from 12 to 18 inches long, with notches cut in its sides, at intervals of a quarter of an inch, to keep the wire in position. To avoid any electromagnetic effect which may be produced by the coils if they happen, when carrying currents, to be placed near the galvanometer, the wire should be doubled on itself at its middle point, the bight put round a pin fixed near one end of the board, and the wire then wound double on the board, the two parts being kept far enough apart to insure insulation. Resistance coils made in this way are exceedingly useful for electric-lighting experiments, as the thickness of the wire and its exposure everywhere to the air prevent undue heating by strong currents, or, if there is much heating, obviate the risk of damage. For the battery a single cell, as for example a gravity-Daniell, or, if the battery is to be carried from place to place, two hermetically-sealed chloride of silver cells, which may be joined in series or in multiple arc as required, may very conveniently be used. Sir William Thomson's graded potential galvanometer * (see Vol. II.) is the most

* That is a high resistance galvanometer in which the needle system, or magnetometer, can be placed with its centre at different distances from the centre of the coil to give different degrees of sensibility, and further provided with one or more magnets to intensify the magnetic

Method of Direct Comparison of Potentials.

convenient instrument (described below) for many practical purposes; but when very great accuracy is aimed at, as when the method is used for the measurement of the specific conductivity of short lengths of thick metallic wires by comparison with a standard, a sensitive reflecting galvanometer of resistance great in comparison with that of the conductor between the points at which the terminals are applied should be employed, and the battery should be of as low internal resistance as possible.

The galvanometer is first set up and made of the requisite sensibility either by adjusting, as described in p. 310 above, the intensity of the field in which it is placed, or, if it is a graded galvanometer, by placing the magnetometer at the position nearest the coil, and dispensing with the field-magnet.

The conductor whose resistance is to be compared, and one of the coils whose resistance is known, are joined in series with the battery. It is advisable to have this circuit at a distance of a few yards from the galvanometer, so that accidental motions of the wires carrying the current may not have any sensible effect on the needle. One operator then holds the electrodes of the galvanometer so as to include between them, say, first the wire which is being tested, then the known resistance, then once more the wire being tested, in every case taking care not to include any binding screw connection, or other contact of the conductors. The known resistance should, when great accuracy is required, be so

field at the needles when required. Sir William Thomson's Graded Galvanometers will be described in Vol. II.

chosen that the readings obtained in these two operations are as nearly as may be equal.

Method of Direct Comparison of Potentials.

Let the mean of the readings for the first and third operations be V scale divisions, for the second V'; let r denote the known resistance, and x the resistance to be found.

Since by Ohm's law the difference of potentials between any two points in a homogeneous wire, forming part of a circuit in which a uniform current is flowing, is proportional to the resistance between those two points, we have,

$$x = \frac{V'}{V} r. \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

The resistance of a contact of two wires whether or not of the same metal may be found in the same manner, by placing the galvanometer electrodes so as to include the contact between them, and comparing the difference of potential on its two sides with that between the two ends of a known resistance in the same circuit. Care must however be taken in all experiments made by this method, especially when the galvanometer circuit includes conductors of different metals, to make sure that no error is caused by thermal electromotive forces. To eliminate such errors the observations should be made with the current flowing first in one direction and then in the other in the battery circuit.

Practical Example.

The following results of some measurements of the resistance of a Siemens $S\ D_2$ dynamo machine, made on May 4, 1883, in the Physical Laboratory of the University

Practical Example.

of Glasgow, may serve to illustrate this method. An iron wire coil, of half an ohm resistance, was joined to one of the terminals of a standard Daniell, and short wires attached to the other terminal of the cell and the free end of the coil were made to complete the circuit through the armature, by being pressed on two diametrically opposite commutator bars, from which the brushes and the magnet connections had been removed. The electrodes of the galvanometer, which was one of Sir William Thomson's dead-beat reflecting galvanometers of high resistance, were applied alternately to the same commutator bars, and to the ends of the half ohm, and the readings recorded. The following are the results, extracted from the Laboratory Records, of three consecutive experiments:

EXPERIMENT I.

Operation.	Reading on Scale.	Deflection of Spot of Light.
Galv. zero read	214	
Electrodes on ½ ohm	857	643
„ „ armature	597	383

EXPERIMENT II.

Galv. zero read	214	
Electrodes on armature	607	393
„ „ ½ ohm	874	660
„ „ armature	607	393

EXPERIMENT III.

Galv. zero read	214	
Electrodes on ½ ohm	874	660
„ „ armature	607	393
„ „ ½ ohm	872	658

The first experiment gives for x the value, $383 \times {\cdot}5/643$, or ·298 ohm. The other two experiments, although

their numbers are different, give very nearly the same result, which agrees closely with a measurement made about eight months before, by the same method, with a graded potential galvanometer. Practical Example.

In the ordinary testing of the armatures of machines by this method, the circuit of the battery may be completed through the brushes; but if the machine has been wound on the shunt system, care must be taken to previously disconnect the magnet coils. In every case the galvanometer electrodes must be placed on the commutator bars directly.

Prof. Tait * has used a differential galvanometer (see below, p. 374) for this method of determining low resistances. The conductors to be compared were arranged in series, so that the same current flowed through both. The terminals of one coil were then placed at two points on one conductor, the terminals of the other coil at two points on the other, such that the galvanometer deflection was zero. The difference of potentials between the points of each pair was therefore the same in the two cases. Hence the lengths of portions of the two conductors of equal resistance were obtained.

Potential Method with Double-arc.

The following zero method, due to Mr. T. Gray, is founded on the same principle. The arrangement of apparatus is shown in Fig. 82. One terminal of a battery of one or two low resistance cells is attached to a stud on a thick copper bar P, the other terminal to a metallic axis round which the copper bar h turns. The bar h makes contact at its outer end with a bare wire and a bare rod bent round into concentric circles with

* *Trans. R.S.E.*, vol. xxviii. 1877–8.

Potential Method with Double-arc.

centre at the axis of the bar, and having a pair of remote extremities connected with mercury cups or binding terminals, and the other pair of extremities free as shown. To one of these terminals is connected one end of the bar to be tested, to the other one end of the standard bar. The other end of one of these bars, say the standard, is connected to a mercury cup S, which is in line with, but is insulated from, a row of mercury cups or a mercury trough cut in a copper bar placed parallel to P. Between this bar

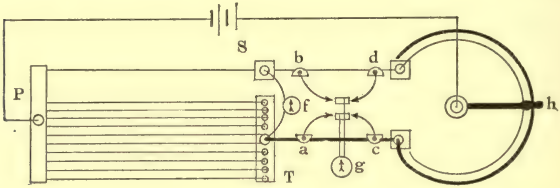

FIG. 82.

and the trough are stretched a series of parallel wires all of the same material and length and as nearly as possible of the same resistance; and a single wire, of the same resistance, material, and length, connects the bar P and the cup S with which the standard bar is in contact. These wires may be conveniently straight rods of platinoid, an eighth of an inch in diameter, and six feet long, soldered at one end to the bar P, and at the other to stout well-amalgamated copper terminals dipping into the mercury cups or trough. The wires may be made of the same resistance by means of a slide wire bridge, or by the method described below.

The cup S and the terminals T are now brought to one potential by turning the bar h round on the circular wire until a sensitive galvanometer, f, joining them shows no deflection. This galvanometer is then left connected, and by means of a second sensitive galvanometer, g, two pairs of points a, d and c, d are found between which in each case no current flows when they are connected by a wire. Each pair of points are therefore at the same potential. Hence if we denote by r_1 the resistance of the standard between b and d, by r_2 that of the other rod between a and c, and by n the number of wires joining P and T, we have

Potential Method with Double-arc.

$$r_2 = \frac{r_1}{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

A differential galvanometer (p. 374 below) with two independent pairs of terminals may be employed for this method. One coil may be made to join a, b, the other c, d, or one coil may be made to join b, d, and the other a, c. In the former case either the effect on each coil must be made zero, or care must be taken to connect the proper terminals to a, b and c, d. The resistance of the galvanometer coils except when the current in each coil is made zero, must be so great as not to cause any sensible alteration of the potentials at the points at which the terminals are applied.

The wires joining P to S and T may be tested for equality as follows. Two nearly equal wires are made to join P to S and P to T, and h is placed so that the galvanometer f shows zero current. The wire joining P to T is then removed and another put in its place.

If the current in f still remains zero for the same position of h the latter wire and the former are of the same resistance. If not the necessary correction is made (see footnote p. 354) and the comparison repeated.

Early Methods of Comparing Resistances.

Before the invention of the Bridge Method resistances were in general compared by a process of substitution; by varying the resistance in a circuit including the resistance to be measured by known amounts; or by means of a differential galvanometer. In the first the resistance to be compared was placed in the circuit of a galvanometer and battery, and the deflection of the galvanometer noted. The unknown resistance was then replaced by a variable and known resistance, which was adjusted until the former deflection was reproduced. The unknown resistance was then taken as equal to the adjusted value of the variable resistance. This method as well as the second involved the assumption that the electromotive force and resistance of the battery remained the same in both experiments. This assumption cannot in general be made with safety if accurate results are required. These two methods are now seldom or never used, and we shall not here further discuss them.

Method by Differential Galvanometer.

The method by differential galvanometer is similar, but does not involve the assumption just referred to. A differential galvanometer is an instrument with two distinct coils each with a pair of terminals of its own. Sometimes the two coils have a common terminal, but this arrangement serves no purpose and renders the instrument inapplicable in many cases. The coils are in general arranged so as to have equal resistance, and

so as to have as nearly as possible equal magnetic effects on the needle for all positions of the latter. Thus in such an instrument when equal differences of potential are established between the pairs of terminals so as to produce currents in opposite directions through the coils, the needle shows no deflection.

Method by Differential Galvanometer.

To determine an unknown resistance r_2 it is joined as shown in Fig. 83 to one of the terminals t, t of a coil

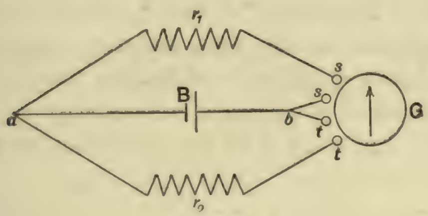

FIG. 83.

of the galvanometer. The other end of the resistance and the other terminal of the coil are connected respectively to the terminals a, b of a battery B. Another resistance r_1, is connected in a similar manner to the terminals s, s of the other coil of the galvanometer, and the terminals a, b of the battery. Thus the battery current flows through the two circuits $b\ s\ s\ a$, $b\ t\ t\ a$, so that the magnetic effects produced at the needle are opposed. The resistance r_1 is adjusted until no deflection is produced. If the resistances of the galvanometer coils including the connecting wires on each side of a, b are equal, r_1 is equal to r_2. It is better however to avoid any error from resistance of connecting wires

by replacing r_1 (which may now be an unknown but conveniently variable resistance such as that given by a rheostat) by such a known resistance r'_1 as does not disturb the equilibrium. Then $r_2 = r'_1$. This result is evidently independent of the electromotive force and resistance of the battery.

Sensibility of Differential Galvanometer Method. To find the sensibility of this method let g_1, g_2 be the resistances of the coils, r the resistance of the battery and wires common to both circuits, r_1, r_2 the remaining parts of the resistances of the respective circuits, γ_1, γ_2 the currents in the coils, m the constant which multiplied into γ_1 gives the deflection produced by the coil of resistance g_1, n the corresponding constant for the other coil, then if a be the deflection, we have

$$a = m\gamma_1 - n\gamma_2 = \frac{E\{m(g_2+r_2) - n(g_1+r_1)\}}{r(r_1+r_2+g_1+g_2)+(r_1+g_1)(r_2+g_2)}.$$

If we denote the denominator of this value of a by D, we may write the equation

$$\frac{Da}{E} = m(g_2 + r_2) - n(g_1 + r_1) \quad . \quad . \quad . \quad (40)$$

Hence if a be accurately zero

$$\frac{m}{n} = \frac{g_1 + r_1}{g_2 + r_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

Suppose, after adjustment has been made as nearly as possible to zero deflection, another resistance r'_1 to be substituted for r_1, then calling D', a', E' the new values of D, a, E, we have by (41)

$$\frac{D'a'}{E'} = m(g_2 + r_2) - n(g_1 + r'_1) \quad . \quad . \quad (40a)$$

Sensibility of Differential Galvanometer Method.

Hence by (40) and (40a)

$$n(r'_1 - r_1) = \frac{D}{E}a - \frac{D'}{E'}a' \quad . \quad . \quad . \quad (42)$$

Here a and a' are so small as not to be observable, and hence to the degree of accuracy to which each deflection is equal to zero $r'_1 = r_1$, whether E be equal to E' or not.

By first adjusting so that a is apparently zero, then putting r_1 out of adjustment by a known amount, and observing the corresponding deflection a', the sensitiveness of the method can be calculated by

$$r'_1 - r_1 = -\frac{D'}{nE'}a' \quad . \quad . \quad . \quad . \quad (43)$$

and E' can be calculated from the known values of the other quantities. If then the smallest observable deflection be known, then E' being taken as constant the corresponding error is approximately known.

In most differential galvanometers $g_1 = g_2$, and $m = n$. In such cases (40a) becomes, the accents being dropped,

$$r_2 - r_1 = \frac{r(r_1 + r_2 + 2g) + (r_1 + g)\,(r_2 + g)a}{nE}$$

Hence if $r_2 - r_1 = \delta r$ where r is a small quantity, we have

$$a = \frac{nE\,.\,\delta r}{(r + g)\,(2r + r_1 + g)} \quad . \quad . \quad . \quad (44)$$

or if r the resistance of the battery be small

$$a = \frac{nE\delta r}{(r_1 + g)^2} \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Equations (44) and (45) give the sensibility of a galvanometer so adjusted.

Best Value of Resistance of Galvanometer Coils.

If it be possible to vary the resistance of the coil, keeping the volume of the conductor constant, then the resistance of each should be made as nearly as possible $\frac{1}{3}$ of the resistance to be measured. For let l = length of wire in each bobbin, a its cross-sectional area, ρ its specific resistance (p. 380 below), v = volume of wire, then $v = la$, $g = l\rho/a = l^2\rho/v$. The value of n is proportional to the length of wire through which the current flows, and may be written kl. We get for (44)

$$a = \frac{klE \, . \, \delta r}{\left(r_1 + \frac{l^2\rho}{v}\right)\left(2r + r_1 + \frac{l^2\rho}{v}\right)}.$$

Let

$$u = \frac{1}{l}(r_1 + g)\,(2r + r_1 + g),$$

where $g = l^2\rho/v$. a will be a maximum when u is a minimum. We have

$$\frac{du}{dl} = \frac{2r}{l^2}(g - r_1) + \frac{r_1 + g}{l^2}(3g - r_1) = 0,$$

and this satisfies the condition for a minimum, viz. $d^2u/dl^2 > 1$.

Hence a is a maximum when

$$g = \tfrac{1}{3}(r_1 + r)\left\{2\sqrt{1 - \tfrac{3}{4}\frac{r^2}{(r_1 + r)^2}} - 1\right\} \quad . \quad (46)$$

If r be small in comparison with r_1, a is a maximum for $g = \frac{1}{3}r$.

A cell or battery of small resistance is generally available, but it is not generally possible to vary g to

any required degree. Differential galvanometers can however be made with their coils in sections, each provided with a pair of terminals so that they can be joined in any possible combination of series or multiple arc.

Method by Differential Galvanometer for Low Resistances.

When the resistances to be compared are small they should be placed across the terminals of the coils of the galvanometer as shunts, and the two shunted coils joined in series with the battery, but in such a way that the currents produce opposite effects on the needle. Let γ be the whole current in the circuit, and as before γ_1, γ_2 be the currents through the coils of resistance g_1, g_2, and let a' in this case be the deflection. Then

$$a' = m\gamma_1 - n\gamma_2.$$

But

$$\gamma_1 = \frac{\gamma\ r_1}{r_1+g_1},\ \gamma_2 = \frac{\gamma\ r_2}{r_2+g_2},\quad \gamma = \frac{E}{r + \dfrac{r_1 g_1}{r_1+g_1} + \dfrac{r_2 g_2}{r_2+g_2}},$$

and therefore

$$a = E\frac{mr_1(r_2+g_2) - nr_2(r_1+g_1)}{D'} \quad . \quad . \quad (47)$$

where

$$D' = r_1 g_1(r_2+g_2) + r_2 g_2(r_1+g_1) + r(r_1+g_1)\,(r_2+g_2).$$

If $a = 0$, $m/n = r_2(r_1+g_1)/r_2(r_1+g_2)$. Hence if $m = n$, $g_1 = g_2$, we have $r_2 = r_1$.

Comparative Sensibility of Method.

To compare when $m = n$ and $g_1 = g_2 = g$, the sensibility of this method with that of the last, we have if $r_2 - r_1 = \delta r$, a small quantity

$$a' = \frac{nEg\ \delta r}{D'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

This method is more sensitive than the last if $D'/g < D$, that is if $r < g$.

The sensibility of the method may easily be found as in the former case.

Measurement of Specific Resistance.

In order that the conducting powers of different substances may be compared with one another, it is necessary to determine their *specific resistances*, that is, the resistance in each case of a wire of a certain specified length and cross-sectional area. We shall here define the specific resistance of any substance at any given temperature as the resistance between two opposite faces of a centimetre cube of the material at that temperature.* This resistance has been very carefully determined for several different substances at ordinary temperatures by various experimenters, and a table of results is given below (Table V.).

To measure the specific resistance of a piece of thin wire, we have simply to determine the resistance of a sufficiently long piece of the wire by the ordinary Wheatstone Bridge method described above, and from the result to calculate the specific resistance. Let the length of the wire be l cms., its cross-section s square cms., and its resistance R ohms. Then the specific resistance of the material would be Rs/l ohms. The length l is to be carefully determined by an accurately graduated measuring-rod; and the area s may be found with sufficient accuracy in most cases by direct

* The reciprocal of this (called below the specific conductivity) may be advantageously called the *electric conductivity* of the substance, if the word conductivity be set free by the adoption of the word *conductance* for the reciprocal of the resistance of a given conductor.

Measurement of Specific Resistance.

measurement, by means of a decimal wire gauge measuring to a hundredth of a millimetre. If, however, the wire be very thin, the cross-section may, if the density is known, be accurately obtained in square cms. by finding the weight in grammes of a sufficiently long piece of the wire (from which the insulating covering, if any, has been carefully removed), and dividing the weight by the product of the length and the density. Very thin wires are generally covered with silk or cotton, which may very easily be removed, without injury to the wire, by making the wire into a coil, and gently heating it in a dilute solution of caustic soda or potash. The coating must not in any case be removed by scraping.

Corrections to be made in Weighing.

If the density is not known, it may be found by weighing the wire in air and in water by the methods described in books on hydrostatics. All the weights, from 1 gramme upwards, ordinarily used in weighing are made of brass, and hence when conductors of nearly the same specific gravity as brass are weighed in air, the correction for buoyancy may be neglected. The weighing in water however must be corrected both for expansion of water with rise of temperature and for the weight of air displaced by the weights. For a temperature of 13°C. these corrections are as follows. For expansion of water an increase of loss of weight in water of ·059 per cent; for buoyancy of air a diminution of apparent weight in water of ·0143 per cent. Care should be taken in weighing to prevent air bubbles from adhering to the sides of the specimen; and the water used for weighing should first have been carefully

boiled to expel the air contained in it. All error of this kind may be avoided by boiling the water with the specimen in it, and then allowing both to cool together.

Measurement of Specific Conductivity of Thick Wire or Rod.

If the wire be thick, and a sufficient length of it to render possible an accurate measurement of its resistance by the ordinary bridge method is not conveniently available, one of the methods of comparing small resistances described above (p. 358 *et seq.*) is to be used. The most convenient in many cases is that just described (p. 366 *et seq.*) in which the resistance between two cross-sections of the bar to be tested is compared with that between two cross-sections of a standard rod of pure copper. The cross-sections should, if the distance between them be not thereby made too small, be chosen so as to make the two resistances nearly equal. If we put V for the number of divisions of deflection on the scale of the potential galvanometer, when the electrodes of the galvanometer are applied to the standard rod, at cross-sections l cms. apart; V' that when they are applied to the rod being tested, at cross-sections l' cms. apart, then we have for the ratio of the resistance of unit length of the standard to the resistance of unit length of the wire tested at the temperature at which the comparison is made, the value $Vl'/V'l$. If s and s' be the respective cross-sectional areas, which in this case are easily determinable by measurement with a screw-gauge, and we assume that the temperature at which the measurements of resistance are made is 0° C., we get for the ratio of the specific resistances at 0° C. the value $V'ls'/Vl's$, and therefore also for the ratio of their

Measurement of Specific Conductivity of Thick Wire or Rod.

specific conductivities $Vl's/V'ls'$. This last ratio multiplied by 100 gives the percentage conductivity at 0° C. of the substance as compared with that of pure copper. If, as will generally be the case, the temperature at which the experiments are made be above the freezing-point, the value of 100 $Vl's/V'ls'$, may be taken as the percentage of the specific conductivity of pure copper at the observed temperature possessed by the substance, and this, if the wire tested is a specimen of nearly pure copper, will be nearly enough the same at all ordinary temperatures.

If in experiments by this method the electrodes are applied by hand to the conductors, the operator should, especially if the electrodes and the conductors tested are of different materials, be careful not to handle the wires, but should hold them by two pieces of wood in strips of paper passed several times round the wires, or by some other substance which conducts heat badly, so that no thermal electromotive force may be introduced into the circuit of the galvanometer (see above, p. 337). He may conveniently make the galvanometer contacts by means of two knife edges fixed in a piece of wood which can be lifted from one conductor to the other without its being necessary to handle the galvanometer wires in any way. This will besides render any measurement of the length of the conductor intercepted between the galvanometer electrodes unnecessary, as l is equal to l'. We have then for the percentage specific conductivity of the substance the value 100 $V's'/Vs$.

As an example of this method we may take the

Practical Example. following results of a measurement (made in the Physical Laboratory of the University of Glasgow) of the specific conductivity of a short piece of thick copper strip. The specimen was joined in series with a piece of copper wire of No. 0 B.W.G. of very high conductivity, in the circuit of a Daniell's cell of low resistance. The electrodes of a high resistance reflecting galvanometer applied at two points 700 cms. apart in the copper wire gave a deflection of 153·5 divisions, when applied at two points 500 cms. apart in the strip 270 divisions. The weight of the wire per metre was 443 grammes, of the strip per metre 186·3 grammes. Hence the specific conductivity of the copper strip was 96·6 per cent. of that of the wire against which it was tested.

Realization of Mercury Standard Ohm. The accurate realization of a standard ohm, as defined on p. 315 above, involves the determination of the specific resistance of mercury, an operation which requires great care and considerable experimental skill. The tube to be used must be very nearly uniform in internal cross-sectional area, and must be filled in a vacuum with perfectly pure mercury. Any deviation of the cross-section from uniformity can be sufficiently nearly allowed for by considering the mercury column as made up of a number of short columns each of cross-section equal to the mean cross-section of that part of the tube. The mean cross-section for each of a sufficient number of such short columns must therefore be determined as accurately as possible by a process of calibration. This is effected by moving a short column of mercury from one position to another along the tube,

and measuring the length which it occupies in each position.

A tube is chosen and is fixed against a finely graduated scale. In order that the tube if accidentally disturbed may be replaced a mark is made upon it, and the position of this on the scale carefully observed and noted; or one end of the tube is made to rest against a piece projecting from the bar to which the scale is attached. If great exactness is desired the scale is engraved on a rigid bar along which move two reading microscopes with cross-wires. By means of these the positions of the ends of the mercury column can be found. For many purposes it is sufficient to read off the positions of the ends of the column by the eye, assisted if need be by a lens, and parallax may be avoided by using a scale graduated on a slip of silvered glass. Calibration of Tube.

Let the column of mercury be approximately $1/n$ of the whole length of the tube where n is a sufficiently great whole number, and let the column be moved in each case from one position to the next through a distance as nearly as may be equal to its whole length, so that n measurements of length are made in the length of the tube. Let r_0, r_1 denote the readings of the ends of the column in the first position, r'_1, r_2, the readings of the ends in the second position and so on, $r_1, r_2, r_3, \ldots r_{n-1}$ being nearly coincident with $r'_1, r'_2, r'_3, \ldots r'_{n-1}$ respectively, while r_0, r_n coincide with the ends of the tube. Now let $r_1 + e_1$, $r_2 + e_2$, &c., be the lengths of the columns of uniform section equal to the mean section of the tube which would have volumes equal to the lengths r_1, r_2, &c. of the actual Calculation of Corrections from Results of Calibration.

Calculation of Corrections from Results of Calibration.

tube. It is the value of the corrections e_1, e_2, &c. which it is the object of the experimental process to discover. Since the successive portions of the tube occupied by the column of mercury are equal in volume, we have for the corrected length of the portion of the tube corresponding to the i^{th} position of the column

$$r_i + e_i - r_{i-1} - e_{i-1} = l \quad . \quad . \quad . \quad . \quad . \quad (48)$$

where l is a constant. Hence putting δ_1, δ_2, &c. for the uncorrected lengths $r_1 - r_0$, $r_2 - r_1$, &c., we get by subtracting $l - e_1 = \delta_1$ from each of the equations of which (48) is the type, the series of equations

$$\left.\begin{array}{l} e_1 - e_2 + e_1 = \delta_2 - \delta_1 \\ e_1 - e_3 + e_2 = \delta_3 - \delta_1 \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ e_1 - e_n + e_{n-1} = \delta_n - \delta_1 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (49)$$

The terminal corrections e_o and e_n are of course zero. Adding, we find

$$e_1 = \frac{\Sigma\delta_i}{n} - \delta_1 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50)$$

where Σ denotes summation for all values of i from 1 to n.

In the same way we get

$$e_2 = \frac{\Sigma\delta_i}{n} - \delta_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

and so on.

If the values of e_1, e_2, &c., be plotted on any convenient scale as ordinates of a curve, the abscissee of which are the corresponding distances measured along the tube

from one end, the curve will give the correction to be used for any other distance measured along the tube.

Control Observations.

This determination is generally sufficient; but if necessary it can be extended as follows: The calibration is performed with a column of double the former length, which is moved along the tube after each observation through a distance equal to half the length of the column. This gives $n-1$ measurements of length. The same process is then repeated with a column of three times the original length, which is moved after each observation through the same distance as before. This gives $n-2$ observations. Then a column of four times the original length is used and so on, until finally a column $n-1$ times the length of the original column is employed, with which of course only two observations are made.

Process of Combining Results.

From the original set of observations we get, if d'_i denote the excess of the observed length of the column in the i^{th} position over that in the $(i-1)$th,

$$\left.\begin{array}{l} e_1 - e_0 = e_2 - e_1 + d'_1 \\ e_2 - e_1 = e_3 - e_2 + d'_2 \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ e_{n-1} - e_{n-2} = e_n - e_{n-1} + d'_{n-1} \end{array}\right\} \quad . \quad . \quad (52)$$

In the same way putting d''_i for the excess of the observed length of the double column in the i^{th} position over that in the $(i-1)$th, we get from the second set of observations

Process of Combining Results.

$$\left.\begin{array}{l} e_1 - e_0 = e_3 - e_2 + d''_1 \\ e_2 - e_1 = e_4 - e_3 + d''_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ e_{n-2} - e_{n-3} = e_n - e_{n-1} + d''_{n-2} \end{array}\right\} \quad . \quad . \quad (53)$$

and so on, the number of equations diminishing by one each time until we get finally from the two last sets of observations

$$\left.\begin{array}{l} e_1 - e_0 = e_{n-1} - e_{n-2} + d_1^{(n-2)} \\ e_2 - e_1 = e_n - e_{n-1} + d_2^{(n-2)} \end{array}\right\} \quad . \quad (54)$$

and

$$e_1 - e_0 = e_n - e_{n-1} + d_1^{(n-1)} \quad . \quad . \quad . \quad (55)$$

Adding all the first equations of the sets (52) &c., then adding all the second equations of the sets and subtracting from the sum the first equation of the first set; next adding all the third equations of the sets and subtracting from the sum the second equation of the third set, and the first of the second set, and so on, we get successively the equations

$$\left.\begin{array}{l} n(e_1 - e_0) = \Sigma d_1 \\ n(e_2 - e_1) = \Sigma d_2 - d'_1 \\ n(e_3 - e_2) = \Sigma d_3 - d'_2 - d'_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ n(e_{n-1} - e_{n-2}) = \Sigma d_{n-1} - d'_{n-3} \ldots - d'_1 \end{array}\right\} (56)$$

where

$$\Sigma d_i = d'_2 + d''_i + \&c.$$

By means of these equations $e_1, e_2, e_3, \ldots e_{n-1}$ can be at once calculated.

Measurement of the Specific Resistance of Mercury.

We give here as an example a short account of an accurate determination of the specific resistance of mercury made by Lord Rayleigh and Mrs. Sidgwick.* Some account of other determinations will be given in connection with the subject of the Realization of Units of Resistance in Vol. II.

The resistances of columns of mercury contained in tubes which had been carefully calibrated were measured by Carey Foster's method as described above. The comparison coils were British Association standards,† preserved in the Cavendish Laboratory, and every care was taken to obtain an accurate result. It was assumed that the resistance R of the column is given to a sufficiently close approximation by the equation ‡

$$R = r\int\frac{dx}{s} \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

where dx is an element of the length of the tube, r the specific resistance of mercury, s the cross-sectional area of the mercury column at the element dx. The value of R was calculated by evaluating the right-hand side of this equation from the results of the calibration as follows. Let λ be the length of the tube occupied by the calibrating column when the middle of its length was at the element dx, then we have for the area of the

* *Phil. Trans. R. S.*, Part I. 1883.

† That is standards constructed according to the determination of the ohm made by the British Association Committee and described in their 1864 report. This ohm is here referred to as the "B.A. unit." Its value is about 1·12 per cent. smaller than the legal ohm. See *Realization of Units of Resistance in Absolute Measure*, Vol. II. below.

‡ See Lord Rayleigh's *Theory of Sound*, Vol. II. § 308.

Measurement of the Specific Resistance of Mercury.

section near this element C/λ, where C is a constant. Putting therefore $s = C/\lambda$, denoting the whole length of the column by L, and the number of points at equal distances apart along the length L, at which the cross-sectional area has been thus found by n, we get by summation

$$R = \frac{rL}{nC}\Sigma(\lambda) \quad . \quad . \quad . \quad . \quad . \quad (58)$$

where $\Sigma(\lambda)$ denotes the sum of the n values of λ.

But if M be the mass of mercury contained in the length of the tube

$$M = \rho\int sdx = \frac{rL}{nC}\Sigma\left(\frac{1}{\lambda}\right),$$

where ρ denotes the density, and $\Sigma(1/\lambda)$ the sum of the reciprocals of the n values of λ. Hence eliminating C between these two equations and solving for r, we get

$$r = \frac{MR}{\rho L^2} \frac{n^2}{\Sigma(\lambda)\Sigma\left(\frac{1}{\lambda}\right)} \quad . \quad . \quad . \quad . \quad . \quad (59)$$

If the tube had been truly cylindrical the value of r would have been $MR/\rho L^2$. The factor $n^2/\Sigma(\lambda)\Sigma\left(\frac{1}{\lambda}\right)$ is the correction for conicality of the tube, and is a numerical quantity a little less than unity.

The tubes used were placed horizontally, and were fitted at each end with hollowed out L shaped pieces of ebonite which formed wide terminal cups. Each end of the tube was fitted air-tightly into the ebonite socket with a short length of thick rubber tubing thrust in a little way, so as to leave room for a caulking of paraffin

wax, which was run, in a melted state, into the mouth of the socket. These cups were filled with mercury and connected with the resistance balance by means of well amalgamated copper rods. The tube was kept at 0° by being immersed in a trough containing melted ice. The temperature of the mercury in contact with the copper rods it was ascertained was not higher than 5° or 6°, and as ice was piled up round the cups, it was estimated that the temperatures of the parts of the tube within the cups, and therefore not directly exposed to the ice-bath, were not higher than 2°.

Measurement of the Specific Resistance of Mercury.

The cups were so large in section that they might be supposed of infinite extent in comparison with the tube, and an addition of ·82 of the diameter * was made to the observed length to correct for the influence of the cups.

The ends of the tube were rounded to a convex form so that the length of the bore could be measured. This was done by adjusting to the ends reading microscopes, fitted with micrometer screws by which the distance between could be varied by an amount known to the $\frac{1}{10000}$ of an inch, reading off on a brass rule substituted for the tube the length to the nearest number of whole divisions, and determining the fractions of divisions at the ends by means of the micrometers.

The tubes were carefully cleaned by passing through them in succession sulphuric acid, nitric acid, caustic potash, distilled water, and finally air dried by chloride of calcium. The calibration was performed by a short

* See above p. 165, and Lord Rayleigh's *Theory of Sound*, Vol. II. § 307.

Measurement of the Specific Resistance of Mercury.

thread of clean mercury, which was moved to the successive required positions by air blown through a chloride of calcium tube; and measured in some cases by substituting an ivory scale under microscopes adjusted to the ends of the column, in others by placing a scale alongside the tube and reading the length off by means of a magnifying glass.

The determination of the mass of mercury contained in the tube, was found by weighing a thread of mercury nearly as long as the tube. The length of this column was found as follows. After the resistance had been measured the greater part of the mercury was retained in the tube, and the ends of the column pressed flat with flat-headed vulcanite pins fitted into the ends of the tube. The length of the column was obtained by the microscopes, and the temperature by a thermometer lying alongside the tube. The mercury was then blown out of the tube and weighed. By comparing the length of the column with the actual length of the tube the whole quantity of mercury contained in the tube at 0° was found with sufficient accuracy.

Experiments were made with four tubes, and the mean value obtained for the resistance at 0° of a column of mercury 1 metre long and 1 square millimetre in cross-section was ·95412 B.A. unit (see footnote, p. 389 above). Previous measurements made by Werner, Siemens, and Matthiessen gave ·9536 B.A. unit and ·9619 B.A. unit respectively for this resistance. It will be noticed that the value just stated lies between these, but much nearer to the former. Messrs. Siemens Brothers for a long time used the resistance of a

column of mercury specified as above as the unit of resistance, and standard units were issued by them to experimenters. One of these examined by Lord Rayleigh gave ·95365 B.A. unit for its resistance at the temperature 16·7° at which it was stated to be correct.

Measurement of the Specific Resistance of Mercury.

For two of the tubes used in these experiments comparisons were made of the resistance at 0° with that in water at the temperature of the room, which was about 13° C. It was found that the mean difference of resistance for 1° per unit of the resistance at 0° was ·000861. This is of course the change of resistance of the contents of a certain glass tube, and not to be confounded with the temperature variation of the specific resistance of mercury.

Forms of the Standard Ohm.

Standard ohms have been made in mercury, by using tubes bent so that the requisite length is obtained in a compact form, but they are not very convenient in use, and are of course liable to breakage. A copy of the standard ohm can however be easily made when the resistance of a column of mercury of definite cross-section and length has been accurately found. Figs. 84 and 85 show such copies. Fig. 84 is the usual form of the standard. It is made of platinum-silver wire, wound within the lower cylinder. The space within up to the top of the wider cylinder is filled with paraffin-wax. The ends of the coil are attached to two thick electrodes of copper rod, bent as shown and kept in position by a vulcanite clamp. The ends of these when the coil is used are placed in mercury cups in the manner already explained, and should always, before the

Forms of the Standard Ohm.

coil is placed in position, be freshly amalgamated with mercury. The lower cylinder up to the shoulder is placed in water when the coil is in use, and the temperature of the water is ascertained by means of a thermometer in the hollow core of the cylinder. The variation of the resistance of the coil with temperature is known, and hence its resistance at any observed temperature can be obtained. Of course care must be

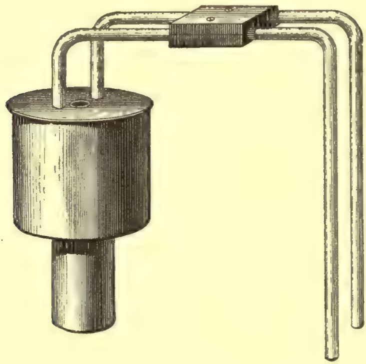

FIG. 84.

taken not to expose the standard to too strong currents, and to keep the temperature as near as possible to the normal temperature at which the standard is given as correct.

Fig. 85 shows a form of the standard constructed by Messrs. Elliott and Co. according to a suggestion made by Professor Chrystal. A thermoelectric couple, of which one junction is within and close to the coil, and

the other outside the case, is used to determine the temperature of the coil. In the form in which the instrument is now made the external junction is not brought out through the bottom of the case as shown, but the wire is brought out at the top of the case, and then joined to a wire of the other metal which is entirely outside and attached to one of the binding

Forms of the Standard Ohm.

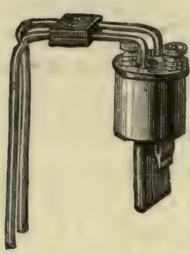

FIG. 85.

screws. The external junction is of course placed in water the temperature of which is measured, and the thermal current is observed by means of a galvanometer connected to the terminals. This gives the difference of temperatures between the junctions and therefore the temperature of the coil.

Measurement of Very High Resistance.

The measurement of a very high resistance such as that of a piece of insulating material cannot be effected by means of Wheatstone's Bridge, and recourse must be had in most cases to electrostatic methods in which the required resistance is deduced from the rate of loss of charge of a condenser, the plates of which are con-

Method by Galvanometer.

nected by the substance in question. If, however, the resistance of the material be not too great, and a large well insulated battery of from 100 to 200 cells, and a very sensitive high resistance galvanometer are available, the following method is the most convenient. First join the galvanometer, also well insulated, and the resistance to be measured (prepared as described below, p. 401, to prevent leakage) in series with as many cells as gives a readable deflection, which call D. Now join the battery in series with the galvanometer alone, and reduce the sensibility of the instrument to a suitable degree by joining its terminals by a wire of known resistance, and, to keep the total resistance in circuit great in comparison with the resistance of the battery, insert resistance in the circuit. Let E and B denote respectively the electromotive force and resistance of the whole battery, G the resistance of the galvanometer, S the resistance joining its terminals in the second case, R the resistance introduced into the circuit of the galvanometer in that case, and X the resistance to be found; we have for the difference of potentials between the terminals of the galvanometer in the first case the value,

$$\frac{EG}{G+B+X} = mD \quad . \quad . \quad . \quad . \quad (60)$$

where m is the factor by which the indications of the galvanometer must be multiplied to reduce them to volts. In the second case the resistance between the galvanometer terminals is $SG/(S+G)$, and therefore the difference of potentials between them is,

Method by Galvanometer.

$$\frac{E\frac{SG}{S+G}}{B+R+\frac{SG}{S+G}} = \frac{ESG}{(B+R)(S+G)+SG} = mD. \quad (61)$$

Hence combining equations (60) and (61) so as to eliminate E and m, and solving for X, we get,

$$X = \frac{D_1}{D}\left(B + R + G + \frac{(B+R)G}{S}\right) - (B + G) \quad (62)$$

If X be great in comparison with the remainder of the resistance in circuit the term $(B + G)$ may be neglected.

Determination of Specific Resistance of Glass.

This method was used by Mr. T. Gray and the author for the determination of the specific resistances of different kinds of glass. The specimens of glass were in the form of thin, nearly spherical flasks about 7 cms. in diameter, with long narrow and thick walled necks. The thin walls of the flask were brought into circuit by filling it up to the neck with mercury, and sinking it to the same level in a bath of mercury, then joining one terminal of the battery to the external mercury by a wire passed down the long neck, and the other to the mercury in the bath without. This mercury bath was an iron vessel contained in a sand bath which could be heated to any required temperature. A well insulated galvanometer (constructed by aid of a grant from the Government Research Fund to a special design described in Vol. II.) of high resistance and great sensitiveness was included in the current. A battery of over

Determination of Specific Resistance of Glass.

100 Daniell's cells was used, and after a reading of the galvanometer in one direction had been taken and recorded, with the corresponding temperature of the glass, the coatings of the flask were connected together until the next reading was about to be taken. For this the current was reversed, and the deflection taken as before, and so on. The "electric absorption" (see p. 435 below) was thus reversed between every pair of readings, and it lasted in most cases about three minutes. The resistances were therefore those existing after three minutes' electrification. The result for the glass of highest insulation tested, which was lead glass of density 3·14, was a specific resistance at 100° C. of about 8400×10^{10} ohms. The resistance was halved for each 8·5° or 9° rise of temperature.

A modification of this method for which one of Sir W. Thomson's Graded Potential Galvanometers or Voltmeters (see Vol. II.) is very suitable, may be used for the determination of the insulation resistance of the conductors in an electric light installation.

Determination of "Insulation Resistance."

The conductors are disconnected from the generator and both ends from one another. They are then joined at one end by the potential galvanometer in series with a battery of as many cells as gives a readable deflection with the magnetometer in the position of the greatest sensibility. The number of divisions corresponding to this deflection is read off, and the number of divisions which the battery gives when applied to the galvanometer alone is then observed. Call the latter number V and the former V'; and let E divisions be the total electromotive force of the battery. Let the resistance

of the battery which may be determined by the method described below (p. 414) be B ohms, the resistance of the galvanometer G ohms, and the insulation resistance to be found R ohms; we have plainly,

Determination of "Insulation Resistance."

$$V = \frac{EG}{B+G}, \quad V' = \frac{EG}{B+G+R}.$$

Therefore,

$$R = (B+G)\left(\frac{V}{V'} - 1\right) . \quad . \quad . \quad . \quad (62)$$

If B be small in comparison with G we may put

$$R = G\frac{V - V'}{V'} \quad . \quad . \quad . \quad . \quad . \quad (63)$$

A shunt-wound generating machine giving sufficient electromotive force may be used instead of the battery, and in this case R is found by equation (63).

The insulation resistance for unit of length is found from this result by *multiplying* by the length of either of the conductors.

This method is applicable to the measurement of the insulation resistance of cables or telegraph lines, but for details the reader is referred to the manuals of testing in connection with these special subjects.

In the case of insulating substances the method just described requires the use of so powerful a battery that it is quite inapplicable except when the specimen, the resistance of which is to be measured, can be made to have a large surface perpendicular to the direction of the

Determination of "Insulation Resistance."

current through it, and of very small dimensions in that direction. Such a case is that of the insulating covering of a submarine cable in which the current by which the insulation-resistance is measured flows across the covering between the copper conductor and the salt water in which the cable is immersed.

In general, therefore, in the determination of the insulating qualities of substances which are given in comparatively small specimens it is necessary to have recourse to the electrometer method mentioned above (p. 395), of which we shall give here a short account.

Measurement of a High Resistance by Method of Leakage.

The most convenient instrument for this purpose is Sir William Thomson's Quadrant Electrometer. For a full description of this instrument, and a detailed account of the mode of using it, see Chap. V. above. The electrometer, having been carefully set up according to the most sensitive arrangement, and found to be otherwise in good working order, is tested for insulation. One pair of quadrants is connected to the case according to the instructions for the use of the instrument, and a charge producing a potential difference exceeding the greatest to be used in the experiments is given to the insulated pair by means of a battery, one electrode of which is connected for an instant to the electrometer-case, the other at the same time to the electrode of the insulated quadrants, and the percentage fall of potentials produced in thirty minutes or an hour by leakage in the instrument is observed. If this is inappreciable, the instrument is in perfect order. For practical purposes the insulation is sufficiently good when the same battery being applied to charge the electrometer alone as is

applied to charge the cable, or condenser formed as described below, there is not a more rapid fall of potentials without the cable or specimen than with it; for there can then be no error due to leakage.

Measurement of a High Resistance by Method of Leakage.

An air condenser, well insulated by glass stems varnished and kept dry by pumice moistened with strong sulphuric acid, is adjusted to have a considerable capacity, and its insulated plate is connected to the insulated quadrants of the electrometer and the other to the electrometer-case to which the other pair of quadrants is also connected. A charge producing as great a potential as in the former case is given to the condenser and electrometer thus arranged, and the fall of potentials observed by means of the electrometer. If the loss in a considerable time be also inappreciable, the condenser insulates properly, and its resistance may be taken as infinite.*

Arrangement of Specimen with Condenser.

The specimen of material to be tested is now placed so as to connect the plates of the condenser. The manner in which this is to be done of course depends on the form of the specimen. If it is a flat sheet, it may be covered on each side, with the exception of a wide margin all round, with tinfoil, and thus made to form itself a small condenser which is to be joined by thin wires in multiple arc with the large condenser. The edges and margins

* A condenser of any other kind, such as those used in cable testing, the insulating material between the plates of which is generally paper soaked in paraffin, may be used instead of an air condenser, but as the resistance of the latter may, if the glass stems be well varnished and kept dry, be taken as infinite, and there is besides no disturbance from the phenomenon called *electric absorption* (see p. 425 below), it is preferable to use an air condenser if possible.

Arrangement of Specimen with Condenser.

of the sides of the specimen should be carefully cleaned and dried, and covered with a thin coating of paraffin to prevent conduction along the surface between the two tinfoil coatings, when the condenser is charged. It is advisable, when possible, to coat the whole surface including the tinfoil with paraffin, and to make the contacts with the tinfoil plates by means of thin wires also coated with paraffin for some distance along their length from the tinfoil. The plate condenser thus formed should be supported in a horizontal position on a block at the middle of the lower surface. The upper coating is made the insulated plate.

If the specimen be cup-shaped, as, for example, if it be in the usual form of an insulator for telegraph or other wires, the hollow may be partially filled with mercury, and the cup immersed in an outer vessel containing mercury, so that the mercury stands at nearly the same level outside and inside. The lip of the cup down to the mercury on both sides is to be cleaned and coated with paraffin, as before, to prevent leakage across the surface. A thin wire connected with the insulated plate of the condenser is made to dip into the mercury in the cup, and a similar wire connected with the other plate of the condenser dips into the mercury in the outer vessel. Strong sulphuric acid may, on account of its drying properties, be used with advantage instead of mercury as here described, when the substance is not porous and is not attacked by the acid.

In every case in which, as in these, the insulating substance and the conductors making contact with it form a condenser of unknown capacity, the condenser

used in the experiment must be arranged to have a capacity so great that the unknown capacity thus added to it, together with the capacity of the electrometer, may be neglected in the calculations. Arrangement of Specimen with Condenser.

The condenser, if it has been disconnected, is again connected as before to the electrometer. One electrode of a battery of from six to ten small Daniell's cells in good order, is also connected with the electrometer case, and the other electrode is brought for a short time, thirty seconds say, or one minute, into contact with the insulated plate of the condenser at any convenient point, such for example as the electrode of the electrometer connected with the insulated pair of quadrants. The battery electrode is then removed, and the condenser and electrometer left to themselves.

The condenser has thus been charged to the potential of the battery, which will be indicated by the reading on the electrometer scale at the instant when the battery is removed. The deflection of the electrometer needle will now fall, more or less slowly according to the insulation resistance of the condenser with its plates connected by the material being tested. Readings of the position of the spot of light on the electrometer scale are taken at equal intervals of time, and recorded, and this is continued until the condenser has lost a considerable portion, say half, of its potential.

Theory of Leakage Method

The resistance of the insulating material is easily calculated from the results in the following manner. Let V be the difference of potentials between the plates of the condenser at any instant, Q the charge of the condenser at that instant, which may be taken as

Theory of Leakage Method.

proportional to the deflection on the electrometer scale, and C its capacity (p. 46). We have $Q = CV$, and therefore $dQ/dt = CdV/dt$. But $-dQ/dt$ is the rate of *loss* of charge, that is, the current flowing from one plate to the other, and this is plainly equal by Ohm's law to V/R. Hence $-dQ/dt = V/R$, and therefore

$$C\frac{dV}{dt} + \frac{V}{R} = 0.$$

Integrating we get,

$$\log V + \frac{t}{CR} = A, \quad . \quad . \quad . \quad . \quad . \quad (65)$$

where A is a constant. If V be the potential difference t seconds after it was V_o, we get by putting $t = 0$ in (65) $A = \log V_o$. Hence (65) becomes

$$\frac{t}{CR} = \log \frac{V_o}{V}$$

and

$$R = -\frac{t}{C}\frac{1}{\log \frac{V_o}{V}} \text{*} \quad . \quad . \quad . \quad . \quad . \quad (66)$$

If $V = \frac{1}{2}V_o$, we have $R = t/\log \frac{1}{2}$.

If the condenser have a resistance so low as to add materially to the rate of discharge, an additional experi-

* It is to be remembered that the logarithms to be here used are Naperian logarithms. The Naperian logarithm of any number is equal to the ordinary or Brigg's logarithm multiplied by 2·302585.

ment must be made in the same way to determine the resistance of the condenser alone, with its plates connected only by its own dielectric. Let R_c denote the resistance of the condenser, determined by equation (66) from the results of the latter experiment, and R_i the resistance of the specimen; by equation (10) (p. 15) $1/R = 1/R_i + 1/R_c$, and therefore

Theory of Leakage Method.

$$R_i = \frac{RR_c}{R_c - R_i} \quad . \quad . \quad . \quad . \quad . \quad (67)$$

If C has been obtained in C.G.S. electrostatic units of capacity, it may be reduced to electromagnetic units by dividing by the number of electrostatic units of capacity equivalent to the electromagnetic unit, that is (see Vol. II.) by 9×10^{20} nearly.

When an air condenser is used, its capacity can generally be obtained approximately by calculation from the dimensions and area of the plates. For example, if two parallel plates of metal, placed at a distance d apart, very small in comparison with any dimension of either surface, have a difference of potentials V, and there be no other conductor or electrified body near, we have seen above (p. 57) that the capacity on a portion of area A near the centre of either plate is $A/4\pi d$. Hence in the example below, we have for the capacity of the disk of area A the value $A/4\pi d$, if we neglect the non-uniformity of the electrical distribution near the edge.

If C has been taken in absolute C.G.S. electromagnetic units of capacity (see Vol. II.), we obtain R

Theory of Leakage Method.

from (66) in cms. per second,* which may be reduced to ohms by dividing by 10^9.

When a condenser such as one of those used in submarine telegraph work is used, the capacity of which is known in microfarads,† then since a microfarad is $1/10^{15}$ C.G.S. electromagnetic units of capacity, we have for R in ohms the formula

$$R = 10^6 \frac{t}{C} \frac{1}{\log \frac{V_o}{V}} \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Practical Example of Leakage Method.

The following are results actually obtained in tests of a specimen of insulating material made in the form of an ordinary telegraph insulator. An air condenser consisting of two horizontal brass disks, the distance of which apart could be regulated by means of a micrometer screw, was joined with the insulator made into a small condenser with mercury inside and outside, as described above. The lower disk was of considerably greater diameter than the upper, which had a diameter of 12·54 cms., and the distance between them was adjusted to be 1 cm. The upper disk was connected to the insulated pair of quadrants, and the lower to the electrometer case. Calling A the area of the upper plate, and d the distance between them, we have, neglecting the effect of the edges of the upper disk, for the capacity of this condenser the value $A/4\pi d$ in C.G.S. electrostatic units. Hence in the actual case $C = 9·828$. The interior surface of the insulator

* In the electromagnetic system of units a resistance has the dimensions of velocity. See Vol. II.

† See Vol. II., also the *Note* in the Appendix to the present volume.

covered by the mercury was so small, and the thickness of the material so great, that, even allowing the material to have a high specific inductive capacity, the capacity of the condenser which it formed was small in comparison with that of the air condenser. The experiment gave, when the condenser and insulator were joined as described, $V_0 = 251$, $V_1 = 100$, $t = 5640$ seconds. Hence,

Practical Example of Leakage Method.

$$R = \frac{5640}{9{\cdot}828 \times 2{\cdot}303 \times \log_e \frac{251}{100}} = 623,$$

in seconds per centimetre (C.G.S. electrostatic units of resistance). As the condenser was not insulating perfectly, a separate test was made for it alone, with the results $V_0 = 239$, $V_1 = 182$, $t = 6120$. Hence

$$R_c = \frac{6120}{9{\cdot}828 \times 2{\cdot}303 \times \log_e \frac{239}{182}} = 2286,$$

and therefore by (67)

$$R_i = \frac{623 \times 2286}{2286 - 623} = 857,$$

in seconds per centimetre.

Multiplying this result by 9×10^{20} (the approximate value of v^2, see Vol. II.), to reduce to electromagnetic units, we get for the resistance of the insulator 7712×10^{20} cms. per second, or 771×10^{12} ohms.

Measurement of Battery Resistance.

The determination of the resistance of an electrolytic liquid is attended with serious difficulty in consequence of the *polarization* in general produced at the surfaces

Resistance of Electrolytes.

Resistance of Electrolytes.

of electrodes in contact with them. This polarization involves in certain cases what has been called a transition resistance at the separating surfaces produced by the presence of the ions or of air, or of both, at these surfaces, and an alteration of the resistance of part of the liquid column due to change in the condition of the liquid near the electrodes. Further it involves an electromotive force opposed to that producing the current, which must be taken account of in most of the ordinary methods of comparing resistances, and this cannot in general be done with accuracy. For example, if V be the difference of potentials between a pair of electrodes in contact with an electrolyte, γ the current through the electrolyte, and E the electromotive force of polarization, we have

$$V = \gamma R + E. \qquad (69)$$

Thus we cannot find R (which after all might not be the true resistance of the electrolyte) by finding V and γ alone; we must find also E. But the value of E depends to a certain extent on the value of γ, and on a variety of other circumstances, such as the size and nature of the electrodes, which render the determination of R by any process of this kind exceedingly difficult.

Polarization Capacity.

The electromotive force of polarization consists in a finite difference of potentials at each electrode. This causes the electrode to act as the plate of a condenser, of which the capacity may be called the polarization capacity of the electrode. This is considerable even for an electrode of very small surface, on account of the

thinness of the stratum at the surface within which the difference of potentials exists.

Non-polarizable Electrodes.

The disturbance from polarization is however small when the liquid is a solution of a metallic salt, and the electrodes are composed of the metal in question. The resistance can then be determined with fair accuracy by the Wheatstone's Bridge, or other ordinary method which may be applicable, if precautions are taken to eliminate any transition resistance which there may be at the plates. It has been found that electrodes of ordinary zinc amalgamated with mercury produce no electromotive force of polarization when placed in contact with sulphate of copper and zinc solution. This fact has been made use of by Beetz,* Paalzow,† and others for the determination of the resistance of zinc sulphate solutions of various strengths. In the experiments of Beetz which were made with great care, the liquid was boiled with the electrodes in position to expel air from the plates and so prevent transitional resistance.

Paalzow's Experiments.

Paalzow also determined the resistances of solutions of other salts by placing the liquid to be experimented on in tubes communicating at their extremities with porous clay cylinders filled with the same liquid and standing in wide glass vessels containing amalgamated zinc electrodes of large surface immersed in zinc sulphate solution. The polarization at the junctions of the two liquids was slight, and, with the resistance of the porous cylinders, was eliminated by observations with columns of different lengths.

* Pogg, *Ann.* cxvii. (1862), p. 1.

† *Ibid.* cxxxvii. (1869), p. 489.

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Ewing and Macgregor's Experiments.

Determinations of the resistance of zinc sulphate and copper sulphate solutions of different strengths have been made by Professors Ewing and Macgregor * by the Wheatstone Bridge method. Two arms of the bridge were made of large resistance, the liquid column (contained in a narrow tube with wide ends, in which were placed platinum electrodes), and the variable resistance were placed in the other two, and a dead-beat galvanometer with a very light mirror and needle used to test for balance. Thus only feeble currents of short duration were sent through the liquid column.

All the reliable experiments on sulphate of zinc agree in showing that for this substance there is a strength for which the specific resistance is a minimum. At temperature 10° C. this strength is by Ewing and Macgregor's experiments approximately that which corresponds to density 1·298.

Horsford's and Wiedemann's Experiments.

Another method for the elimination of polarization was first used by Wheatstone † and more lately by Horsford and by Wiedemann.‡ A measurement was made of the apparent resistance for one length of the liquid column, then the column was shortened by moving the electrodes closer together, and the current through the column restored to its former value by adding wire resistance. The amount of resistance thus added gave the resistance of the portion of the column removed from the circuit. The state of the electrodes cannot however here be taken as absolutely the same in

* Trans. R.S.E. vol. xxvii. (1873), p. 51.

† Phil. Trans. R.S., 1843 ; Scientific Papers, p. 122.

‡ Pogg, *Ann.* xcix. (1856).

any two experiments. In Wiedemann's experiments the electrodes were silver for silver solutions, copper for copper solutions, and platinum in other cases.

Method of Comparison of Potentials.

A method preferable to any of these consists in an application of the potential method described above for the measurement of wire resistance. Contact is made by means of platinum electrodes at two cross-sections of the liquid column at a definite distance apart, while a steady current is kept flowing along the column. The difference of potentials between these electrodes is measured by means of a suitable electrometer, and compared with that between two points in a wire of known resistance through which the same current is flowing. The effect of any electromotive force independent of the current may be eliminated by taking the observations for both directions of the current.

It is here necessary that the capacity of the part of the electrometer charged by the contact be small in comparison with the polarization capacity (p. 408 above) of the electrodes, otherwise the charging current would give a sensible polarization effect at the electrodes. The capacity of the quadrants of a quadrant-electrometer is sufficiently small to avoid any serious error from this cause with electrodes of ordinary platinum wire.

Since the value of the electromotive force of polarization is small when the value of the current is small it is possible to use a high resistance galvanometer instead of an electrometer in this method. It is necessary however to have the current exceedingly small; and

Method of Comparison of Potentials.

therefore if a sensitive electrometer is available it is preferable to use it.

This method seems to have been first used by Branly* in some measurements of the Electromotive Force of Polarization, but it has occurred to and been used by several other experimenters.

Some other methods of determining the resistance of an electrolyte, which involve electro-magnetic considerations, will be given in Vol. II.; and the subject of Polarization will be more fully considered in connection with the Determination of Electromotive Forces. The foregoing sketch may be supplemented by a reference to Wiedemann's *Lehre von der Elektricität,* Bd. I., §§. 563—608, in which will be found a very full account of experiments and results.

Battery-Resistance.

We shall now consider very briefly the measurement of the resistance of a battery. This term is not perfectly definite in meaning, as there is reason to believe that the resistance as well as the electromotive force of a battery depends to some extent on the current flowing through the battery, and further the resistance and the electromotive force, and possibly also the polarization of the battery are affected by differences of temperature. But the information which in practice we generally require from the test, is really what available difference of potentials can be obtained with a certain working resistance in the external circuit. This could be obtained at once by connecting the terminals of the battery by this resistance, and measuring the difference

* *Compt. Rend.* lxxiv. (1878), p. 528.

Battery-Resistance.

of potentials by means of a quadrant electrometer or a potential galvanometer. If we call this difference of potentials V, and the electromotive force of the battery when on open circuit E, then putting R for the external resistance we may write

$$\frac{E}{R+r} = \frac{V}{R} = \gamma \quad . \quad . \quad . \quad . \quad (70)$$

where r is a quantity the definition of which is simply that it satisfies this equation. If the battery had the same electromotive force E, when generating the current C, as when on open circuit, then r would be the effective resistance of the battery; but, although this is not the case, we may without being led into error still speak of it as the resistance of the battery for the current γ. In fact, the value of r, thus found for a particular value of R, does actually enable us to calculate from the known electromotive force for open circuit, with a moderate degree of approximation in the case of a constant battery, and also, but less surely, in the case of a secondary battery, what available difference of potentials will exist between the terminals of the battery when connected by other and somewhat widely differing values of R, and therefore also to find what arrangement of a battery it will be best to adopt in any given circumstances. So far as this practical result is concerned, the numerous methods which have been devised for the determination of the resistance of a battery before any sensible polarization (which requires time to develop) has been set up are, though interesting in

Battery-Resistance.

themselves, of no practical value, and we shall not here describe any of them.

From equation (70) we have

$$r = \frac{E - V}{V} R \quad . \quad . \quad . \quad . \quad . \quad (71)$$

To determine r therefore we have simply to measure with a potential galvanometer the difference of potentials which exists between the terminals of the battery when on open circuit, or connected only by the galvanometer coil, the resistance of which we suppose to be very great in comparison with r, and again to measure in the same way the difference of potentials when the terminals are connected by a resistance R, also small in comparison with that of the galvanometer.*

If the galvanometer scale be graduated so that readings are proportional to the tangents of the corresponding angles, we have, if D be the deflection in the first case, and D' the deflection in the second case, the equation

$$r = \frac{D - D'}{D'} R \quad . \quad . \quad . \quad . \quad . \quad (72)$$

Instead of a potential galvanometer a quadrant electrometer may be employed if the battery is not too large, and the same formula applies when D and D' are taken proportional to the tangents of the angles through which the mirror is turned.

A resistance coil, which may be of German silver

* If the battery consist of a large number of cells, it may be divided into sections and so tested, or each cell may have its resistance measured separately.

wire, constructed as described in p. 367, should be used for the resistance connecting the terminals, and if the current passing through it be considerable its resistance should be determined when the current is flowing. This may be done by including in its circuit a current-galvanometer, and determining the current γ through the wire in amperes,* when V is read off in volts * on the potential instrument. The resistance of the wire with that of the current-galvanometer is in ohms V/γ, and this is to be used as the value of R in equation (72). Battery-Resistance.

If a galvanometer of high resistance be not available, an approximate test can be made by means of a sensitive galvanometer of low resistance. The battery and galvanometer are joined in series with a resistance R, and again with a resistance R'. Let D and D' be the deflections, which must have a difference comparable with either. Then, supposing E and r to be the same in both cases, and putting G for the resistance of the galvanometer we have

$$D = m\frac{E}{R + G + r}, \quad D' = m\frac{E}{R' + G + r},$$

where m is a constant.

Therefore we find

$$r = \frac{D'R' - DR}{D - D'} - G \quad . \quad . \quad . \quad . \quad (73)$$

Mance has shown how to determine the resistance of a battery by means of Wheatstone's Bridge. The battery is placed in the position BD of Fig. 70 above, and a key is connected between B and C. The resist- Mance's Method for the Resistance of a Cell.

* See Vol. II., also the *Note* in the Appendix to the present volume.

Mance's Method for the Resistance of a Cell.

ances r_1, r_2, r_3 are adjusted until the depression of the key produces no alteration in the galvanometer deflection. The galvanometer and the key, with their respective connecting wires, are then conjugate conductors (p. 159 above); and it is easy to show that the resistance of the battery is then r_2r_3/r_1. The needle of the galvanometer is kept nearly at zero by means of a small magnet during the adjustment of the resistances, so that it is as sensitive as possible to any alteration of current produced by depressing the key.

This method is so troublesome as to be practically useless, chiefly on account of the variation of the effective electromotive force of the cell produced by alteration of the current through the cell which takes place when the key is depressed. Prof. O. J. Lodge * has discussed the method, and shown how it may be improved by inserting a condenser in series with the galvanometer between C and D. Still it is inconvenient and gives no information which may not be obtained more easily in another way, and we shall therefore not enter into further detail regarding it.

Thomson's Method for the Resistance of a Galvanometer.

Sir William Thomson † has however shown how the same mode of operating may be made to give the resistance of a galvanometer when there is no other galvanometer available. The arrangement of Fig. 70 is varied by placing the galvanometer in the position BD of Fig. 70, and a key in the position there shown as occupied by the galvanometer. The deflection of the galvanometer produced by depressing the battery key is

* *Phil. Mag.* 1877, p. 515.

† *Proc. R.S.* Vol. xix. (Jan. 1871).

Thomson's Method for the Resistance of a Galvanometer.

nearly annulled by means of a magnet, and the resistances r_1, r_2, r_3 are adjusted until no alteration of the galvanometer deflection takes place when the key in CD is depressed. When this is the case C and D are at the same potential, since the addition of the conductor CD does not disturb the current distribution in the network; and we have for the resistance r_4 of the galvanometer

$$r_4 = \frac{r_2}{r_1} r_3.$$

CHAPTER VIII.

COMPARISON OF CAPACITIES AND MEASUREMENT OF SPECIFIC INDUCTIVE CAPACITY.

SECTION I.

COMPARISON OF CAPACITIES.

Measurement of Electrostatic Capacity.

THE determination of the electrostatic capacity of a condenser is effected by a process in which its charge at a given potential is compared with that required to charge a standard condenser to the same potential. The standard condenser is generally one of which the capacity can be found by calculation from the dimensions and arrangement of the instrument, or which has been itself compared with such a condenser.

Different Forms of Standard Condenser.

There are three forms of standard condenser, the capacity of which can be determined with accuracy by calculation from the geometrical arrangement. There are—

1. Spherical Condensers.
2. Guard-ring Condensers.
3. Cylindrical Condensers.

Faraday's Form of Spherical Condenser.

The simplest form of spherical condenser consists of two spherical conducting surfaces concentric with one another and separated by a dielectric. Such a condenser

Faraday's Form of Spherical Condenser.

was used by Faraday in his experiments on Specific Inductive Capacity, and is shown in Fig. 86. An outer brass shell B is supported on a base-piece as shown in the figure, and is fitted above with a tubulure r, filled by a long plug of shellaç b. The internal brass ball A is supported in a position concentric with the outer

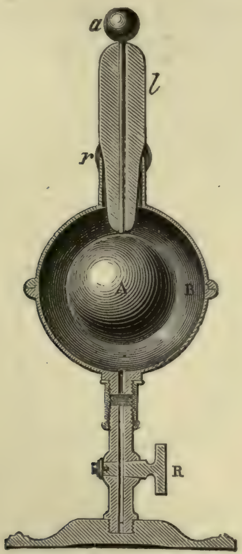


FIG. 86.

shell by a thin stem passing up through the shellac plug and terminating in a knob a. The support below is perforated so as to form a tube by which the space between the spheres can be filled with dry air or any gas. A stopcock r enables this passage to be closed.

This condenser was not used by Faraday for the measurement of capacities in absolute measure, but two of them were employed in the manner described at p. 452 below for the determination of specific inductive capacities. An absolute condenser on this principle has however been constructed by Sir William Thomson, and is shown in section in Fig. 87. The radius of the internal sphere was 4·511 centimetres, of the inner sur-

Thomson's form of Spherical Condenser.

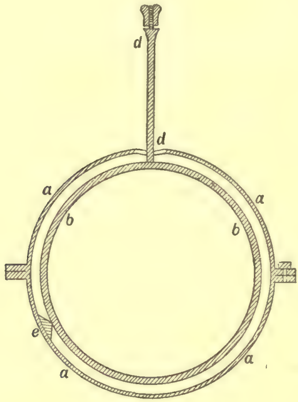


FIG. 87.

face of the external shell 5·857 centimetres. The inner shell was supported in its place by three pieces of vulcanite, of which one is shown in the figure, and communication was made with the interior conductor by a wire passing through the centre of a circular orifice cut in the outer shell. Calculating the capacity of this condenser by (56) Chap. I. above we get $k = 63·264$ centimetres. It was found however

that ·255 centimetre had to be added to this number to correct for the effect of the support and the conducting wire. It is difficult to make the surfaces of such a condenser truly spherical, and to fix them so accurately in their places as to enable the capacity to be calculated with sufficient exactness, and comparisons of this condenser with others showed that this value of the capacity was probably too low. A preferable condenser is therefore Sir William Thomson's guard-ring form of

Thomson's Guard-ring Condenser.

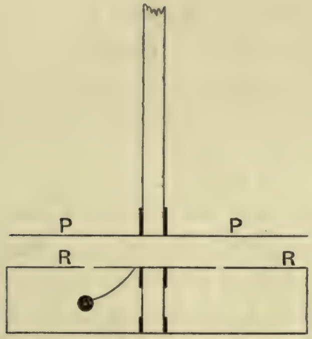


FIG. 88.

the parallel plate condenser. This is shown diagrammatically in section in Fig. 88. (An actual instrument constructed by Dr. J. Hopkinson is shown in Figs. 101, 102 below.) The guard-ring *R* forms as it were part of a cylindrical metal box nearly closed by the disc *D* which the ring surrounds. This box and disc are supported on a glass stem well covered with clean shellac, and a separate glass stem within the box insulates the disc *D* from the ring. A wire passing through a hole

Thomson's Guard-ring Condenser.

in the cylindrical wall of the box makes contact with the electrode of the disc. The other plate of the condenser is formed by the large disc P above. This plate is carried by a glass stem mounted in a socket at the extremity of a fine screw working in a fixed nut above. By turning the micrometer head of this screw, the distance of P from the opposite disc can be altered by any required amount. The condenser and its supporting framework are mounted on an iron sole-plate, round which is cut a circular groove to receive a protecting glass cover; and to enable a dry atmosphere to be maintained about the insulating stems, fragments of pumice moistened with strong sulphuric acid are contained in a lead tray placed on the sole-plate.

Mode of using Guard-ring Condenser.

The manner of using the condenser is as follows: The guard-ring and disc are connected together and charged to the potential required, while the opposite plate is kept at zero potential. The disc is next disconnected from the guard-ring, which is then brought also to zero potential. The charge which was formerly on the disc remains upon it, and since the distribution was very nearly uniform the capacity can be calculated, and therefore the charge on the disc, from the previously existing potential. The effective area of the disc may be taken as the arithmetic mean of the actual area of the disc and that of the opening in the guard-ring. If S be this mean area we have by (61), p. 57 above,

$$C = \frac{S}{4\pi d},$$

and therefore for the charge Q upon the disc when the condenser is charged to potential V

$$Q = \frac{VS}{4\pi d}.$$

A cylindrical condenser of variable capacity has also been invented by Sir William Thomson, and used by Messrs. Gibson and Barclay in their determinations of the specific inductive capacity of paraffin described

Thomson's Sliding Cylindrical Condenser.

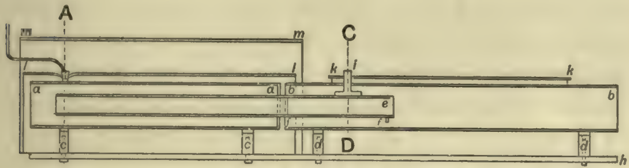


FIG. 89.

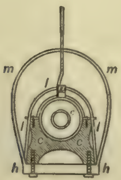


FIG. 90.

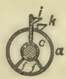


FIG. 91.

below. The instrument is represented in longitudinal section in Fig. 89, and in cross-section through C and A in Figs. 90 and 91. The essential parts are two circular cylinders of brass *aa*, *bb* of the same diameter, supported, with their axes in line and a gap between their adjacent ends, on vulcanite pieces *cc*, *dd*, attached

Thomson's Sliding Cylindrical Condenser.

to a sole-plate *hh*. The lengths of these cylinders were 26·58 centimetres and 35·3 centimetres respectively, and their common diameter 4·9674 centimetres. These dimensions were determined by a measurement of the volume of water contained by the tubes and an accurate determination of their lengths. A third brass cylinder *ee* was supported coaxially within the other two, on four vulcanite feet near one end resting on the inner surface of the outer cylinder. The length of this cylinder was 36·6 centimetres, and its diameter (found by winding fine wire round the cylinder, measuring the length of a certain number of turns, and allowing for the thickness of wire and the spiral arrangement) was 2·303 centimetres. This last cylinder is loaded so as to rest stably on its supports, and can be slided backwards or forwards in the direction of its length so as to alter the relative lengths of it enclosed within the two tubes *aa*, *bb*. A vertical arm *g* projects upwards through a slot cut in the tube *bb*, and carries an index which moves along a graduated scale *kk*. This scale was graduated into 360 divisions, each 1/40 inch or ·0635 centimetre nearly.

A cylinder of metal *ll* fastened to the base of the instrument surrounds the other tube *aa*, to protect it from external influence, and the whole is enclosed within an outer case *mm*.

In the use of the instrument the tube *bb*, the internal cylinder *ee*, and the outer cylinders *ll*, *mm* were connected to earth, while *aa* was insulated and charged. The theory of the instrument is given at p. 61 above. According as the capacity of the condenser was to be

increased or diminished, *ee* was slided towards the left or right, and the amount of change of capacity was given by using the displacement l, measured on the scale *kk*, in the formula

Thomson's Sliding Cylindrical Condenser.

$$C = \frac{1}{2} \frac{l}{\log \frac{r'}{r}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $r' = 2{\cdot}4837$, $r = 1{\cdot}1515$. The capacity when $l =$ one scale division $= {\cdot}0635$ centimetre, was therefore ${\cdot}0413$ centimetre.

This instrument has been modified so as to give it greater range by adopting the arrangement shown in Fig. 92. Here both *ee* and *ll*, (*b* and *c* of the figure) are movable, so as to alter the capacity of *aa*.

Second form of Sliding Condenser.

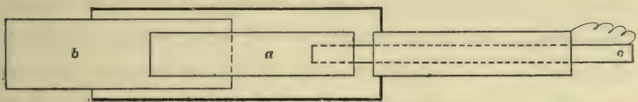


FIG. 92.

Except when the dielectric is a gas, the phenomena of charge and discharge are complicated, and the results of experimental comparisons of the capacities of condensers more or less affected, by what is generally called *Electric Absorption.* If a condenser having a solid or liquid dielectric be charged by applying a battery for a time sufficient to give a uniform potential V throughout the charged plate of the condenser and then be left to itself, its potential will be found after the lapse of a short time to have considerably

Electric "Absorption."

Electric "Absorption."

diminished. This diminution of potential is only partly due to conduction through the dielectric or to want of proper insulation. Part of it is due to a change produced in the dielectric medium when the condenser is charged, which requires time to bring it about, and is called electric absorption from the original idea that it was caused by the penetration of part of the electric charge into the substance of the dielectric. A further charge is necessary to restore the former potential, and if this be given by a second short application of the original charging battery, a second fall of potential not so great as the first will be produced from this cause, and so on for a third, fourth, fifth, &c., short application. Thus if the condenser be charged by a long-continued application of the battery, it will take a considerably greater charge than if the same potential had been produced by an instantaneous or short-continued application. Similar results are obtained when a condenser is discharged. If it has been charged by a long contact with the charging battery, or has been left to itself for some time after charge by a short contact, and is then discharged by a short contact, it will be found immediately after to be at zero potential, but after some little time it will be found again to have acquired a potential of the same sign as before, and can be again discharged. In this way three or four or more discharges can be obtained before its plates are permanently reduced to zero potential. These discharges after the first constitute what is called the residual charge of the condenser.

The phenomena of residual charge have been a good

deal investigated of late years. Kohlrausch * first pointed out the close connection between the phenomena of residual charge and the slow working out of subpermanent strain shown by many elastic substances, and called by German physicists *Elastische Nachwirkung.*† It has been found for example by Dr. Hopkinson that if a Leyden jar be charged positively by an application of a battery continued for a long time, say a week, then negatively for a shorter time, say a day, then positively for a very much shorter time, say a few minutes, the residual discharge will be alternately positive and negative. This behaviour is closely analogous to that of a wire which has been held twisted for different intervals in successively opposite directions. Dr. Hopkinson has also found that mechanical agitation of the dielectric such as that produced by tapping the jar has a marked effect in accelerating the residual discharge.

Residual Charge.

Attempts have been made with fair success, notably by Clerk-Maxwell, to account for electric absorption by imagining the dielectric to be heterogeneous, in the sense of being made up of different imperfectly insulating substances, such that the ratio of the specific inductive capacity to the specific conductivity is not the same for the different media.

* Kohlrausch has shown that the instantaneous discharge is independent of the residual charge, and that for a given jar left to itself for a given time after charging, the residual charge is proportional to the initial potential.

† *Pogg. Ann.* 91, 1854. See also on this subject *Encyc. Brit.* Art. 'Electricity,' by Prof. Chrystal: Ayrton and Perry, *Viscosity of Dielectrics,* Proc. R.S. 1878.

Residual Charge.

It might appear from the preceding that owing to the existence of electric absorption the capacity of a condenser is an indefinite quantity, depending on the time of charge or discharge. This is not the case however, as it has been found by several experimenters that for ordinary condensers, provided the time of charge or discharge do not exceed an interval of a quarter or half a second, the charge required to produce a potential V, or which is withdrawn in annulling a potential V, are sensibly the same and independent of the duration of the contact. This is called the instantaneous charge of the condenser, and the capacity of a condenser is defined as the amount of the instantaneous charge required to produce unit potential at its insulated coating, while the other is at zero. The methods of comparing capacities described below will not therefore (except in the case of cables which require a sensible time to acquire throughout the same potential) involve any ambiguity.

Thomson's Platymeter.

In the investigation of the specific inductive capacity of paraffin referred to above, the capacities of two condensers were compared by an instrument invented by Sir William Thomson, and called by him a platymeter. This instrument is represented in Fig. 93. A brass cylinder *cc*, 22·94 centimetres long, and 5·1 centimetres in diameter, is supported by vulcanite pieces *dd*, and coaxial with it are placed in symmetrical positions, and insulated by the vulcanite supports *ee*, two equal shorter cylinders of thin brass, each 7·68 centimetres in length and 8·6 centimetres in diameter. p, p' thus form corresponding plates of two nearly equal cylindrical con-

densers, of which the opposite plates are furnished by the cylinder *cc*. The whole is enclosed within a metal case *mm*, through which pass insulated by plugs of paraffin the electrodes *qq* of p, p', and the electrode n of *cc*.

Comparison of Capacities. 1. Method by Platymeter.

The platymeter was used with the sliding condenser in the following manner for the determination of the capacities of other condensers. The cylinder *aa* of the sliding condenser was connected to p', the insulated plate of the condenser to be measured to p, and the other plate and cylinders *bb*, *ee* to the case of a quadrant

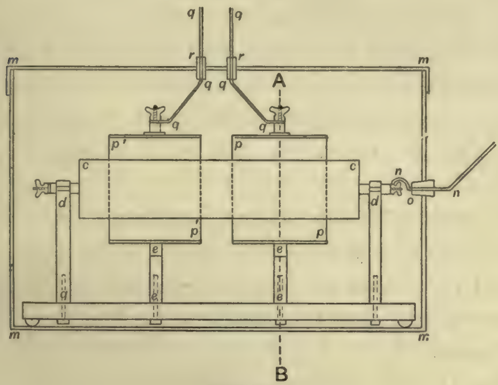


FIG. 93.

electrometer arranged for heterostatic use. The inner cylinder *cc* of the platymeter was connected to the electrode of the insulated pair of quadrants. We shall denote the condenser to be measured and the sliding condenser by A and B, their respective capacities by C, C', and the nearly equal capacities of p, p' respectively

Comparison of Capacities. 1. Method by Platymeter.

by c, c'. Now suppose a positive charge given to A, and the electrodes of the electrometer connected for an instant to reduce the potential of the cylinder cc to zero, and p and p' then connected so as to share the charge on A and p with B and p'. Assuming the action between p and cc to be equal to that between p' and cc, that is, the two sides of the platymeter to be precisely equal, it is plain that the resulting potential of cc must be positive, zero, or negative according as the capacity $C + c$ is greater than, equal to, or less than $C' + c'$. It is plain also that, under the same conditions, the potential of cc must be negative, zero, or positive when B is the positively charged conductor, or positive, zero, or negative, if B be negatively charged. In Gibson and Barclay's experiments one conductor was positively, the other negatively charged, as this gave more marked effects without increased risk of breaking down of insulation.

The capacity of the sliding condenser was adjusted so that when A was connected to p' no alteration in the potential of cc was produced by putting pp' in contact after charging. On the assumption that $c = c'$, this gave $C = C'$.

Comparison of Two Capacities by Imperfect Platymeter.

It was found however that when A and B were interchanged without alteration of their capacities the connection of p with p' disturbed the potential of cc. The two sides of the platymeter were therefore not exactly equal. But in order that the potential of cc should be unaltered after the two condensers are put into contact, it is only necessary that their capacities should be adjusted so as to be in the ratio of the

capacities of the sides of the platymeter with which they are respectively in contact. The capacity of the sliding condenser in the interchanged arrangement was therefore altered until the effect of making contact was rendered zero. Calling the new capacity C'_1, we have the two equations

Comparison of Two Capacities by Imperfect Platymeter.

$$\frac{c}{c'} = \frac{C}{C'}, \qquad \frac{c}{c'} = \frac{C'_1}{C},$$

and therefore

$$C = \sqrt{C' C'_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

As an example we may take the measurement of the capacity of the sliding condenser when the index was at a given position of the scale. This was done by comparing it with the spherical condenser already described. The sliding condenser was adjusted so that when connected to the side p of the platymeter, and the spherical condenser to p', the potential of cc remained unchanged when after the system was charged as described, p and p' were put into contact. The reading on the scale of the sliding condenser was then 211. The condensers were then interchanged and the same operations repeated, and the reading 183 was obtained on the sliding condenser. A second pair of experiments gave 211 and 186 as the readings.

Example: Comparison of Sliding with Spherical Condenser.

Now the capacity of the sliding condenser per scale division was found, p. 425, to be ·0413 centimetre. Hence taking the value 63·519 centimetres for the capacity of the spherical condenser, its capacity in terms of that corresponding to a scale division of the sliding condenser taken as unit was 1538. Calling the

Example: Comparison of Sliding with Spherical Condenser.

capacity of the sliding condenser when the slide was at zero, A, we have for the total capacities of the sliding condenser in the first pair of experiments $A + 211$ and $A + 183$, and in the second pair $A + 211$ and $A + 186$. Hence taking the arithmetic mean instead of the geometric, we have approximately

$$A = 1538 - 198 = 1340,$$

and for the capacity C in C.G.S. units

$$C = 1340 \times \cdot 0413 = 14\cdot 04.$$

2. Cavendish's Method: Comparison of Two Guard-ring Condensers.

The following method given by Maxwell for the comparison of the capacities of two guard-ring condensers, is a modification of a method used by Cavendish for the approximate comparison of two parallel plate condensers of the simpler form. The reader can easily make a diagram for himself by drawing diagrammatically two guard-ring condensers side by side. Let A, B, C denote respectively the small disc, guard-ring with metal backing, and large disc of one condenser, A', B', C' the corresponding parts of the other condenser. The following operations are performed while B is kept connected to C', and B' to C, all connections being made with wires of negligible capacity.

1. A is connected to B and C', and with the electrode J of a Leyden jar or a large battery, and A' is connected to B' and C, and with the earth.

2. A, B, C' are insulated from J.

3. A is insulated from B and C', and A' from B' and C.

4. B and C' are connected with B' and C and with the earth.

5. A is connected with A'.

6. A and A' are connected with the electrode of the insulated quadrants of an electrometer or with a sensitive electroscope.

Comparison of Two Guard-ring Condensers.

By this process A and A' are charged to equal and opposite potentials, and if their capacities are equal the resulting potential after operation 5 is performed will be zero, and the electroscope will show no deflection. By adjusting therefore one of the condensers until this result is obtained the capacity of the other condenser can be found in terms of that of the first. Thus the effect of putting a slab of some insulating substance between the plates of one of the condensers can be determined by performing this process before and after the introduction of the slab. All the operations here described can be performed in rapid succession by a properly arranged and well insulated key.

If the condensers be not guard-ring condensers this method can yet be applied with accuracy in any case in which A and A' may be regarded as surrounded by the other plates C and C'. For example A may be the insulated cylinder aa of a sliding condenser, and A' the internal surface of a spherical condenser, or with sufficient accuracy the interior coating of a Leyden jar. It is only necessary in the above operations to regard B as coincident with C', and B' with C.

3. Faraday's Method.

The following method is practically that used by Faraday in his determination of specific inductive capacity. Two condensers have their plates, which are usually uninsulated, connected to earth, and one of the other plates is charged to a potential which is observed

Faraday's Method.

by means of an electrometer. The insulated plate of the other condenser is then brought into contact with the charged plate by means of a fine wire, and the diminished potential is observed by the electrometer. If one of the condensers is an air condenser, that should be the condenser first charged, and the contact with the insulated plate of the other should be made only for an instant and then broken. This avoids the phenomenon referred to above as electric "absorption" which takes place in solid dielectrics. Calling C_1, C_2 the capacities of the condensers, c that of the part of the electrometer charged by being put in contact with the condenser, V the potential before and V' that after the sharing of the charge, then since the charge remains constant we have

$$V(C_1 + c) = V'(C_1 + C_2 + c) \quad . \quad . \quad . \quad (3)$$

If c is negligible as it generally is this gives

$$\frac{C_2}{C_1} = \frac{V}{V'} - 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Faraday compared the potentials V, V' by bringing a carrier ball into contact with the knob of the condenser before and after the discharge, and comparing by the torsion balance the charges carried off in the two cases (see below p. 452).

Elimination of Capacity of Electrometer.

If the capacity c of the electrometer is not negligible, then if it be supposed independent of the deflection, another equation may be found with which to eliminate it, by first charging the electrometer to some potential V, and then sharing the charge with

the condenser of capacity C_1 so as to give a potential V'. This gives

$$vc = v'(C_1 + c).$$

Hence substituting in (3) above we get

$$\frac{C_2}{C_1} = \frac{V - V'}{v'} \cdot \frac{v}{v - v'} \quad . \quad . \quad . \quad . \quad (5)$$

We shall now describe some methods of comparing capacities which are useful in cable testing, and in the determination of the capacities of condensers in cable work generally. 4. Thomson's First Null Method.

The first of these methods, which is due to Sir William Thomson, requires three condensers of known, one of them of variable, capacity, besides the condenser the capacity of which is to be measured. Let the four condensers be called A, B, C, D, their capacities be denoted by C_1, C'_1, C_2, C'_2, and let C be the variable condenser and D that of which the capacity C'_2 is to be found. (A figure may be made by the reader.) The insulated plates of A, C are first connected together and brought to some convenient potential by giving them a charge from a Leyden jar, or by applying one terminal of a battery the other terminal of which is connected to the earth. They are then disconnected, the charged plate of A put in contact with the insulated plate of B, and that of C with the insulated plate of D. An electrometer of which both pairs of quadrants are insulated, has one electrode connected to A and B, and the other to C and D, and C is varied in capacity, if need be, until both pairs of condensers are brought to

Thomson's First Null Method.

the same potential, which will of course be the case when the deflection of the electrometer has been reduced to zero. We have if V be the potential of A and C before contact with B and D, and V' the common potential after the adjustment has been made

$$V' = \frac{VC_1}{C_1 + C'_1} = \frac{VC_2}{C_2 + C'_2},$$

or

$$C'_2 = \frac{C'_1}{C'_2} C_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

A well insulated and sensitive galvanometer with insulated key may be arranged instead of an electrometer between the pairs of charged plates, and the criterion of equality of potentials will then be zero deflection of the galvanometer needle when the key, previously kept raised, is tapped down after the operation described above. The use of a galvanometer has however the disadvantage that the whole series of operations must be gone through at each discharge. This is not necessary when an electrometer is used, as then only potentials are compared without discharge.

If D be a condenser of great capacity, such as a long cable with the further end insulated in air, time must be given for the condenser to become charged throughout to the same potential, and a corresponding time for the equalization of the potential of D with that of C when these condensers are put in contact. The time generally allowed for a long cable is twenty to thirty seconds and about the same for equalization.

In order to ensure accuracy the condensers C_1, C_2,

C'_1, C'_2 should be all, if possible, nearly equal. In any case C_1 should not be small in comparison with C'_1, nor C_1 in comparison with C_2.

5. Thomson's Second Null Method.

The next method is also due to Sir William Thomson and is much used in cable testing. The arrangement of apparatus is shown in Fig. 94.

A battery of, say, twenty Daniell's cells, insulated by having for the outer containing vessel dry vulcanite or earthenware pots supported on a dry table or board, has

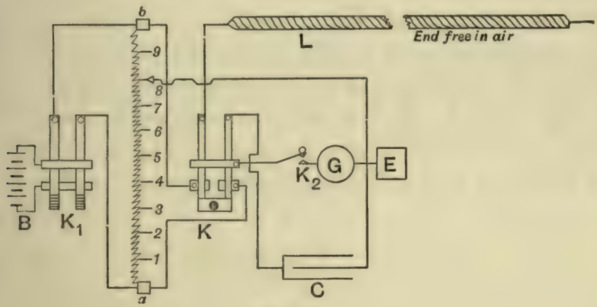


FIG. 94.

its terminals connected through the reversing key K, to the extremities of the series of resistances a, b. These resistances are connected at equal intervals as shown diagrammatically with pieces of metal, which form a set of contact pieces, along which a slider carrying a binding-screw can be moved as in the instrument described above (p. 322), and so the resistance between the slider and the extremities of a, b, varied. A wire attached to the slider is connected to earth, to which are also connected the uninsulated coatings of

Thomson's Second Null Method.

the condensers C and L to be compared. C is here supposed to be the standard or known condenser, L a cable with its remote end free in air. The terminal a of the resistance slide is connected with the insulated coating of the condenser L, the terminal b with the insulated coating of C through the insulated key K. This key besides being capable of giving these connections, can also be made to disconnect the resistance slide from the condensers, and to put the insulated coating of the condensers into contact. By being brought into contact with a and b the respective condensers are charged to the potentials of those points. Now since the slide is at zero potential, if V_1 be the potential of A, R_1 the resistance between A and the slider, and R_2 the resistance between the slider and b, the potential at b will be $-V_2$ where

$$-\frac{V_1}{V_2}=\frac{R_1}{R_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Hence the potential of the condenser L is $-V_2$ and that of C is V_1, and these potentials are proportional to the respective resistances R_1, R_2. By means of the key K the condensers are brought to one potential, and this is zero if $V_1C_1 = -V_2C_2$. To test whether the potential is zero, the key K_2 is depressed and connects the insulated coatings of the condensers to earth through a sensitive galvanometer G. Any difference of potentials between the coatings and the earth is thus annulled and gives rise to a current through the galvanometer. The slider is adjusted until no current

is thus produced through the galvanometer. We have then

$$-\frac{V_2}{V_1}=\frac{R_2}{R_1}=\frac{C_2}{C_1},$$

or

$$C_2=\frac{R_2}{R_1}C_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

For accuracy R_2 and R_1 should be somewhat high resistances so as to ensure an exact knowledge of their ratio, and C_1 should be as nearly as possible equal to C_2.

When a cable is tested sufficient time must be given in charging to enable it to acquire the same potential throughout, and for the discharge of one condenser into the other; and the tests are repeated with the battery reversed on the slide to eliminate the effect of any existing charge in the cable. It is usual also to make a number of tests and take the mean result. Application to Cable.

Instead of a more or less elaborate key K arranged to perform all the operations quickly and conveniently, a system of two pairs of cups 1, 2, 3, 4 arranged in the square order

1	2
3	4

may be cut in a slab of paraffin and filled with mercury. The terminals of a, b are connected to 1, 2, the insulated plate of the condenser to 4, and that of C to 3. By a connecting bridge of wire held by an insulating handle,

Application to Cable.

1 and 3 are connected, and in the same way 2 and 4, so as to charge the condensers. These connections are then removed, and 3 and 4 connected so as to discharge one condenser into the other. Then by means of the key K_2, or by another mercury cup, connected by a wire bridge with 3 or 4, the condenser coatings are connected with earth through the galvanometer.

Plainly in this case also an electrometer may be used instead of the galvanometer. One pair of quadrants is connected to earth, the other pair through the key K_2 to the condensers.

6. De Sauty's Method.

The following method of comparing capacities, which is due to Mr. de Sauty of the Eastern Telegraph Company, is convenient for the comparison of the capacities of condensers in which electric absorption does not come into play. The arrangement of the apparatus is shown in the diagram, Fig. 95. K is a key which when depressed puts into contact with the point of junction of two variable resistances R_1, R_2, one terminal a of a battery, the other terminal b of which is connected to the earth. The other extremities C, D, of these resistances are connected to the insulated coatings of the condensers C_1, C_2, which are to be compared. The other coatings of these condensers are connected to earth. C and D are connected likewise through a sensitive galvanometer G. When the key K is not depressed it joins A directly through a wire to the earth. R_1, R_2 are adjusted so that neither in charging the condensers by applying the battery to A, nor in lischarging by allowing the key to connect A directly to earth, does any current pass through the galvano-

meter. (If any influence of electric absorption is sensible, the ratio of resistances which gives zero galvanometer current when charging will not generally be the same for charge as for discharge.) When no deflection of the galvanometer needle takes place, the potential at C and D must throughout the discharge have been the same at each instant, for the condensers De Sauty's Method.

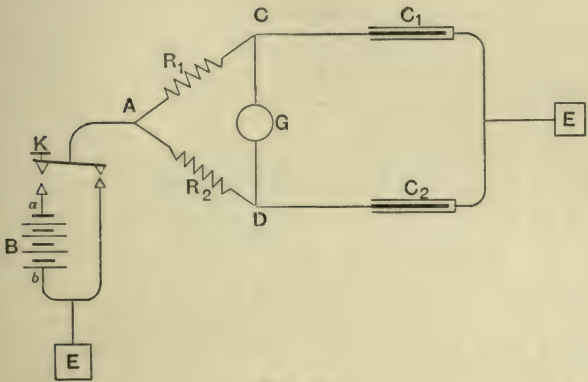


FIG. 95.

could not discharge in such a way as to give a current, first in one direction, then in the other, through the galvanometer, and so keep the needle at rest. But if γ_1 be the current through R_1 and γ_2 the current through R_2, V the common potential of C and D, C_1, C_2 the capacities of the condensers connected with R_1, R_2 respectively, we have

$$\gamma_1 = \frac{V}{R_1} = \pm\frac{d(VC_1)}{dt}, \quad \gamma_2 = \frac{V}{R_2} = \pm\frac{d(VC_2)}{dt},$$

De Sauty's Method.

and therefore

$$R_1 \frac{d(VC_1)}{dt} = R_2 \frac{d(VC_2)}{dt},$$

that is the products of R_1, R_2 into the time rates of variation of the charges of the corresponding condensers are equal at each instant. This can only be the case if

$$R_1 C_1 = R_2 C_2,$$

or

$$C_2 = \frac{R_1}{R_2} C_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

This result may be seen more easily as follows. Let n equal condensers have their insulated coatings joined to A by wires of equal resistance in the manner shown for two condensers in Fig. 93. Then plainly the charging or discharging current in each wire will be the same at each instant, and the insulated plates will always be at one potential. No change will be caused by joining the insulated coatings in two groups by wires of zero capacity, so as to make the groups virtually two condensers, of capacities equal in each case to the sum of the capacities of the separate condensers of the group, and connected to A by wires of resistances inversely as the capacities. By making n sufficiently large, and the capacity of each condenser sufficiently small, the capacities of the groups may be made of any required value and nearly enough in any ratio commensurable or incommensurable.

7. Direct Deflection Method.

Another method, which we shall again refer to later as a method of obtaining the capacity of a condenser

in absolute units, is frequently employed to obtain rapidly a comparison of the capacities of two condensers. It is called the Direct Deflection Method. One of the condensers is charged to a measured potential and then discharged by connecting it to earth through a "ballistic" galvanometer, that is a galvanometer (see Vol. II.) the needle system of which has a considerable moment of inertia. Fig. 96 shows the

Direct Deflection Method.

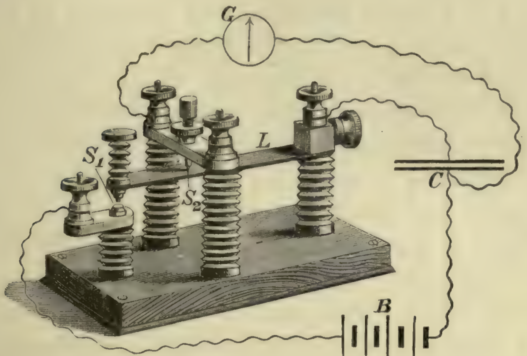


FIG. 96.

arrangement of apparatus with a form of charge and discharge key, the contact pieces of which are mounted on ebonite pillars to ensure high insulation. The spring lever L is provided with two platinum contacts opposite to the platinum pieces S_1, S_2. When depressed it makes contact for charge, when released it connects the plates of the condenser through the galvanometer. If the duration of discharge is, as it generally is, short, and means are taken, for example, by depressing the key

Direct Deflection Method.

immediately after the discharge contact, to disconnect the galvanometer immediately after the first discharge so as to avoid any effect of residual discharge due to electric absorption, the discharge may be regarded as having wholly taken place before the galvanometer needle has moved from zero. The total deflection of the needle from zero is observed. By placing the galvanometer between the battery and S_1, the deflection produced by charging can be observed. If there is leakage this latter deflection will obviously be greater than the former; and if the leakage be not too great the mean of the two deflections with the same battery may be taken as giving the capacity of the condenser. The other condenser is now charged to a potential V' and discharged in the same manner through the galvanometer and the deflection again observed. V and V' should if possible be chosen so as to make the two deflections nearly equal, in order to eliminate the damping effect which the needle experiences to different degrees in deflections of different amounts. If an instrument for comparing the potentials V, V' is not available, they may be produced by applying to the condensers one terminal of a well-insulated battery, the other terminal of which is connected to the earth, and varying the number of cells until equality of deflections is nearly obtained. If the battery be composed of similar cells in good order, the potentials may be taken as proportional to the number of cells applied to produce them. For a rough determination it is convenient of course to charge both condensers by the same battery, and thus to the same potential, and to

take the capacities as proportional to the galvanometer deflections produced.

8. Sir W. Siemens' Method of Comparing a Large with a Small Condenser

The capacity of a large condenser, such as a long submarine cable with its conductor insulated, may be compared with that of a relatively small condenser by the following method, which is due to the late Sir W. Siemens. Let the large condenser be charged to any convenient potential V by means of a battery. If the capacity be C the charge is VC. Now let the large condenser be connected to the insulated coating of the small condenser, the capacity of which we shall suppose to be c. The common potential of the two condensers will now be $VC/(C+c)$. Now disconnect the small condenser and discharge it, and again connect it to the large condenser, disconnect and discharge as before. The potential will now be $VC^2/(C+c)^2$. Thus after n applications in this manner of the small condenser to the large, the potential of the large condenser will be $VC^n/(C+c)^n$. The deflection on a ballistic galvanometer produced by the nth discharge of the small condenser is now noted. The small condenser is then charged, by the same battery as that used to charge the large condenser, and therefore to the same potential V, discharged, and the deflection noted. If D_n, D be these deflections we have

$$\frac{D}{D_n} = \frac{(C+c)^n}{C^n},$$

and therefore

$$C = \frac{c\sqrt[n]{D_n}}{\sqrt[n]{D} - \sqrt[n]{D_n}} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Sir W. Siemens' Method of Comparing a Large with a Small Condenser.

The comparison by this method must be made as rapidly as possible in order that the effect of any leakage of the large condenser may be made as small as possible. On the other hand the theory of the method proceeds on the assumption that the potential of the condenser at each discharge is brought throughout to the same value, and this cannot be done in a long cable unless a sufficient time of contact is given at each discharge. There is further the difficulty of correcting the deflections for air damping, &c. The method therefore cannot be regarded as an accurate one for the cable application.

It is easy, when the ratio C/c is approximately known, to investigate the best value of n to use to give results as little as possible affected by errors in the observation of D, D_n, but on account of the inaccuracies inherent in the method for most practical purposes, it is of little importance to use that value.

9. Sir W. Siemens' Method by Slow Discharge through High Resistance.

The arrangement described above (p. 400) for the determination of a high resistance gives also a means of determining the capacity of a condenser. For let the coatings of the condenser be connected by a very high known resistance R as described, and let a difference of potential V between the coatings be produced by applying a battery. Let V be observed by means of an electrometer, the insulated quadrants of which are kept connected to the insulated coating of the condenser. As the charge diminishes by conduction through the resistance, the electrometer shows a diminishing deflection which is observed at accurately noted instants of time. If V_0, V be the potentials at the beginning

and end of an interval of t seconds, C the capacity of the condenser, and R the resistance connecting the coatings, we have

Sir W. Siemens' Method by Slow Discharge through High Resistance.

$$C = \frac{t}{R} \frac{1}{\log \frac{V_0}{V}} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Values of V_0, V, for different values of t are given by the observations, and enable a mean value of C to be obtained free to some extent from errors of observation.

The resistance R must of course be very great in order that the whole charge may not be so quickly lost as to prevent the potentials from being observed before and after a sufficiently long interval of time. If the condenser be not a perfectly insulated air condenser, the actual resistance of the dielectric layer between its coatings may be taken advantage of, and will in general be convenient for the purpose. To determine it we use an auxiliary condenser of known capacity C', and resistance R', which has been determined by some method, for example, the method of p. 405 above. The insulated coating of this condenser is joined to that of the condenser to be measured, so that the capacity of the joint condenser becomes the sum of their separate capacities, and the resistance between their coatings $RR'/(R + R')$. The condenser thus formed is charged and the potential at different instants of time observed as before. Thus if V_0', V' be the potentials before and after an interval of t' seconds, we have

$$C + C' = \frac{t'(R + R')}{RR'} \frac{1}{\log V_0/V'} \quad . \quad . \quad (12)$$

This equation with (11) suffices to determine C and R.

10. Method by Successive Charge and Discharge. We give lastly here a method of measuring capacities, which was used by Dr. Werner Siemens, and is of importance in the determination of Specific Inductive Capacities. Fig. 97 shows the arrangement of the apparatus. B is a battery of a number of well insulated constant cells, of which one terminal is connected to

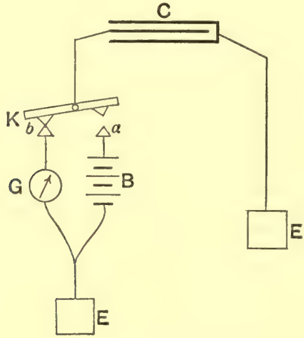


FIG. 97.

earth, and the other terminal to the contact piece a. A sensitive galvanometer G has one terminal connected to the contact piece b, and the other connected to earth. The pieces a and b are so arranged that a commutator, represented diagrammatically by K, connected permanently with the insulated coating of the condenser A, makes contact alternately with a and b. The condenser is charged when a is in contact, discharged when b is in contact. This is easily managed by means of a rotating cylinder carrying contact pieces which are

pressed on by springs represented by a, b, or by some other suitable mechanical arrangement. The commutator is made to give a constant and large number n of discharges, say from 40 or 50 per second upwards. Thus if the battery remains constant, a constant mean current is produced through the galvanometer. Let E be the electromotive force of the battery, and C the capacity of the condenser, then on the supposition that we may suppose the condenser completely charged or discharged at each contact, we have for the mean current nEC. If n be sufficiently great this will give the same deflection as a continuous current of the same amount. After this deflection has been observed, the circuit of the battery is completed through the galvanometer, and a resistance R, just of sufficient amount to give a second good measurable deflection of the galvanometer needle. If α, β be these deflections corrected so as to be proportional to the mean current (generally the actually observed deflections may be taken if they are small), we have

10. Method by Successive Charge and Discharge.

$$nEC = m\alpha$$

$$\frac{E}{R} = m\beta$$

where m is a constant.

Hence we have

$$C = \frac{\alpha}{\beta}\frac{1}{nR} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The commutator may be easily arranged so as to charge the condenser alternately positively and nega-

Method by Successive Charge and Discharge.

tively. If C be the mean of the two capacities which the condenser has, according as one or the other coating is made the uninsulated coating, we have, putting n for the *number of reversals* per second, $2nE'C$ for the whole quantity of electricity which flows through the galvanometer in a second, that is, the mean current. Hence if a and β have the same meanings as before, we have

$$C = \frac{a}{\beta}\frac{1}{2nR} \quad \ldots \ldots \quad (14)$$

These values of the current, it is to be remarked, are obtained on the assumption that the times of charge are sufficiently long to allow the condenser to be fully charged to potential E, and the time of discharge also long enough to allow the condenser to be completely discharged. The results of experiments made with different time-intervals have justified this assumption for small condensers even for time intervals so small as $\frac{1}{20000}$ of a second.

The methods of comparing capacities which depend more or less on electromagnetic principles will be described in Vol. II.

SECTION II.

MEASUREMENTS OF SPECIFIC INDUCTIVE CAPACITY.

ALL measurements of Specific Inductive Capacity involve in practice a comparison of the capacity of a condenser with air as the dielectric with that of the same condenser with the whole or part of the space between the plates occupied by the substance of which the specific inductive capacity is to be found. For practical purposes the specific inductive capacity of air (which is nearly the same at all ordinarily attainable temperatures and pressures) at 0° and under standard atmospheric pressure (760 mm. of mercury) is usually taken as unity, and it will be convenient at present to follow this custom.

Relation of Sp. Ind. Cap. to Index of Refraction.

According to the Electro-Magnetic Theory of Light (see Vol. II.), the specific inductive capacity of a dielectric should be equal to the square of the index of refraction μ_∞ of the medium for light waves of infinite length.* This index is usually calculated from the measured values of the index for known wave lengths by the formula $\mu_\infty = A + B/\lambda^2$, where λ is the wave length. It is however to be noted that this is a formula

* Strictly $\mu^2_\infty = \kappa \times$ magnetic permeability, or magnetic inductive capacity, of the medium. But there is no transparent dielectric for which the magnetic permeability differs much from that for air, which is here taken as unity (see Vol. II.).

of extrapolation, and that the value which it gives may very frequently be seriously in error. The values of μ_∞ thus calculated are given below in some cases for comparison; in others the value of μ for the line D is given.

Cavendish's Experiments.

The first measurements of this kind were made by Cavendish,* by a method the same in principle as that described above, p. 432. He found for glass a mean value of about 8·22, for shellac 4·47, and for wax 4·04. These values later experiments have shown to be too great, no doubt in great measure from the effects of electric absorption.

Faraday's Experiments.

Faraday's experiments were made by the method and apparatus sketched at pp. 433 and 419 above. Two condensers of the form shown at p. 419, and as nearly equal as possible, were constructed. The inner surface of each had a diameter of 2·33 inches, and the outer shell of each an internal diameter of 3·57 inches. To test the equality of the condensers the following process was employed. The condensers were set at some little distance apart, so that the inductive influence of one on the other might be neglected, and in positions such that they were as nearly as possible similarly placed with respect to all external conductors, including the observer. The external coatings were then connected once for all to the earth. The interior coating A of one condenser was then charged, while that of the other, B, remained uncharged. The potential of A was then tested by bringing a small carrier ball into contact with the knob, and observing the force produced at a given distance on

* *Elect. Res.* p. 144, *et seq.*

the suspended ball of a torsion balance. To observe the rate of loss of charge the observations were repeated after a short interval, and the result showed only a slight dissipation. The charge of A was then shared with B by bringing A and B symmetrically into contact by their knobs. The potentials of B and A thus produced were then tested by the carrier ball as before, the charge from B being taken by the ball at the instant of contact with A. The following are two sets of results. The numbers are degrees of torsion of the glass thread of the balance and may be taken as proportional to the charges.

Verification of Equality of Two Condensers.

I. Centres of Balls in Balance 160° apart.		II. Centres of Balls in Balance 150° apart.	
A	B	A	B
	0		152
254			148
250			
Charge divided.		Charge divided.	
	122	70	
124			78
Both discharged.		Both discharged.	
1			5
	2	0	

Thus, taking the experiment I., the charge divisible between A and B may be taken as 249. As B was found immediately after discharge with 122 it may be taken as having received that amount at least. The

Verification of Equality of Two Condensers.

other may be taken as having retained 124. These numbers do not differ much from 124·5, the half of the disposable charge. Again taking experiment II., the disposable charge on B may be taken as 143, and the amount of this given to A is 70, and the amount retained 73. These numbers are again nearly equal to half the disposable charge 71·5, and the discrepancy is in the opposite direction. Hence the capacities of A and B may be regarded as very nearly equal.

To make sure that the instrument would plainly show changes of capacity, Faraday put a metallic lining into the lower hemisphere of one of the instruments so as bring down the distance between the internal ball and the outer coating to ·435 inch. A comparison of the capacities of the condensers made by the same process as before gave 1·08/1 as the ratio in which the capacity of the condenser had been increased. The true ratio was more nearly 1·2/1. But the result showed that a real alteration of capacity of the condenser could be unmistakably recognized in spite of the unavoidable errors of experiment.

Determination of Sp. Ind. Cap. of Shellac.

Having thus satisfied himself of the sensibility of his apparatus, Faraday introduced a thick hemispherical cup of shellac into the lower hemisphere of one of the equal condensers, and compared the capacities in the manner described above, by first charging one and then sharing the charge with the other and observing the reduced potential immediately after. Each of the apparatus was made in turn the condenser to be first charged. The following are the results of such an experiment:

Determination of Sp. Ind. Cap. of Shellac.

I.		II.	
A (Shellac).	B (Air).	A (Shellac).	B (Air).
0			0
	304	215	
	297	204	
Charge divided.		Charge divided.	
113			118
	121	118	
Both discharged.		Both discharged.	
0			0
	7	0	

Calling C' the capacity of the shellac condenser, C that of the air condenser, V the potential before and V' the potential after the sharing of the charge, we have by (4) above

$$C' = \frac{V - V'}{V'} C.$$

Hence from experiment I. we get

$$C' = \frac{290 - 113 \cdot 5}{113 \cdot 5} C = 1 \cdot 55\ C \text{ nearly},$$

and from experiment II.

$$C' = \frac{118}{204 - 118} C = 1 \cdot 37\ C \text{ nearly}.$$

The much smaller result in the second case is due to dissipation and absorption in the shellac condensers between the instant at which the reading 204 was obtained and that of the division of the charges. Faraday estimated the corrected result as nearly $1 \cdot 47\ C$.

Determination of Sp. Ind. Cap. of Shellac.

From four experiments made by this method Faraday obtained a mean result of $1\cdot5\ C$ for the capacity of the shellac condenser. Now plainly, if we regard the direction of the lines of force in the space between the coatings as everywhere radial, that is, neglect the curving down towards the shellac of lines starting from the lower part of the upper hemisphere of the inner ball, we have denoting by K the specific inductive capacity of shellac relatively to air

$$\frac{1+K}{1+1} = \frac{C'}{C} = 1\cdot5,$$

or

$$K = 2.$$

Sp. Ind. Cap. of Glass, &c.

In the same way Faraday found for flint glass $K = 1\cdot76$, for sulphur $K = 2\cdot24$, and for spermaceti that K was between $1\cdot3$ and $1\cdot6$. For oil of turpentine and naphtha he obtained results which indicated a higher specific inductive capacity than that of air, though here the results were rendered uncertain by the influence of conduction.

A long series of experiments was also made by Faraday on different gases, and it was found that so far as the means of measurement went all had the same specific inductive capacity, and that this was independent of temperature and pressure.

For further information as to Faraday's experiments the reader is referred to the original memoirs.*

* *Exp. Res.* Series XI. p. 371, *et seq.*

The specific inductive capacity of paraffin was determined by Messrs. Gibson and Barclay * in 1870, using the platymeter and sliding condenser described above. The paraffin condenser compared is shown in Fig. 98. *aa* is a cylindrical brass vessel 15·5 centimetres deep, and 8·61 centimetres in diameter. At the bottom of

Gibson and Barclay's Experiments on Paraffin.

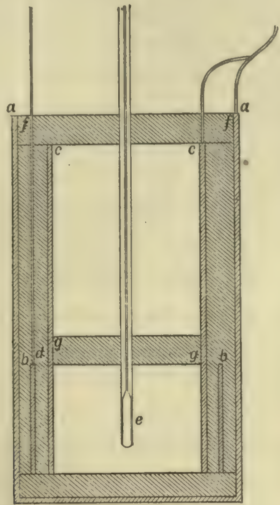


FIG. 98.

this cylinder is a layer of paraffin 1 centimetre thick. On this layer rests coaxial with the outer cylinder, a brass tube *bb*, 4·3 centimetres long, 7·2 centimetres in internal diameter, and ·115 centimetres thick. Inside *bb* and coaxial with it is a cylinder *cc*, 13·1 centimetres

* *Phil. Trans.* 1871, p. 573.

Gibson and Barclay's Experiments on Paraffin.

long and 6·1 centimetres in external diameter. The space between *aa* and *cc* was filled up with paraffin, and from the imbedded tube *bb* an electrode *dd* of fine wire was led to the outside.

The condenser thus formed was placed in an outer vessel containing water of which the temperature was given by a thermometer. A second thermometer fixed in a paraffin plug *ff*, resting on *cc*, gave the temperature of the interior. The paraffin plug *gg* inserted at the level of the top of *bb*, together with *ff*, prevented the passage of heat between the interior of *bb* and the air above the condenser.

The outer vessel *aa*, and the tube *cc* were connected with the earth, and the inner tube *bb* to one side of the platymeter, and balance obtained against the sliding condenser as described above, p. 429. Taking the capacity of the sliding condenser as 1384 times that for each scale division, which it now was in consequence of a small addition which had been made to its value at zero, the mean of a large number of experiments gave for the value of that of the paraffin condenser 1684 times the same unit, or an absolute capacity of 69·552 C.G.S. electrostatic units. These experiments, which were made at different temperatures, showed no alteration of specific inductive capacity with change of temperature. The capacity of the same condenser with the paraffin between the cylindrical plates removed was found in the same way to be 35·394 C.G.S. units, but this was subject to a correction for the cake of paraffin which was left at the bottom to support *bb* and *cc*. The final result was that for paraffin $K = 1{\cdot}977$.

Sp. Ind. Cap. of Paraffin.

Boltzmann's Experiments on Solids: 1. Method by Condenser.

Some very important determinations of specific inductive capacities have been made by Boltzmann.* In his first series of experiments he determined the value of K for ebonite, paraffin, sulphur, and rosin. The method was a modification of that of Cavendish referred to above. A parallel-plate air condenser, the plates of which were supported on insulated stems carried by sliding pieces movable along a graduated horizontal bar, and so could be placed at different measurable distances apart, had one plate connected to earth while the other plate was charged by means of a battery. Different battery-powers of from 6 to 18 Daniell's cells were used in the experiments. After the condenser had been thus charged, the charge was shared with the insulated quadrants (formerly at potential zero) of a Thomson's electrometer, the capacity of which had been increased by means of a small air condenser.

The potential after the charge was thus shared, and while the condenser was still connected, was observed. A direct application of the battery to the electrometer gave in the same way the previous potential of the condenser.

The addition of the small condenser to the electrometer rendered the united capacities of the electrometer and small condenser nearly the same for all deflections, leaving only an increase of capacity of about 1/5 per cent. for each 100 divisions of deflection from zero. This was to some extent eliminated by a double set of observations, first as just described, then by connecting

* *Wien. Ber.* 66, 67 (1872, 3.)

Boltzmann's Experiments on Solids: 1. Method by Condenser.

the condenser for the sharing of the charge, and the battery when applied direct, for so short a time that the charging was over before the needle had appreciably moved. As however the error from this source could hardly be greater than the inevitable inaccuracies in a determination of this kind, we shall here neglect it.

If c be the capacity, assumed constant, of the electrometer and added condenser, C_1 that of the sliding condenser, V_1 the potential before, and V'_1 the potential after the charge was shared, d_1 the distance between the plates, supposed so close that the effect of the edges may be neglected, we have by (4) above

$$C_1 = c\frac{V_1 - V'_1}{V'_1} = \frac{m}{d_1} \quad . \quad . \quad . \quad . \quad (15)$$

where m is a constant.

In order to make the results depend not on the absolute distance between the plates, but on the much more accurately measurable difference of two distances, a similar set of observations was made, still with air only between the coatings, but with another distance d_2. Calling the capacity C_2, the potentials V_2, V'_2 in this case, we have

$$C_2 = c\frac{V_2 - V'_2}{V'_2} = \frac{m}{d_2} \quad . \quad . \quad . \quad . \quad (16)$$

A disc of the substance, the value of K for which was to be found, somewhat larger than the plates of the condenser, was placed in a parallel position between them, so that the induction between the plates took

place everywhere across the disc. The same process was followed, and gave potentials V_3, V'_3 for a distance d_3, and a thickness of disc e. Hence if C_3 be the capacity of the condenser

Boltzmann's Experiments on Solids: 1. Method by Condenser.

$$C_3 = c\,\frac{V_3 - V'_3}{V_3} = \frac{m}{d_3 - e + \dfrac{e}{K}}. \quad . \quad . \quad (17)$$

Putting $C_1 = 1/\lambda_1$, $C_2 = 1/\lambda_2$, $C_3 = 1/\lambda_3$, we get from equations (15) and (16) $m = (d_1 - d_2)/(\lambda_1 - \lambda_2)$ and hence from (17) $\lambda_3 = (\lambda_1 - \lambda_2)\,(d_3 - e + e/K)/(d_1 - d_2)$. Hence remembering that $\lambda_1(d_1 - d_2)/(\lambda_1 - \lambda_2) = m\lambda_1 = d_1$, we have finally

$$K = \frac{e}{\dfrac{\lambda_3 - \lambda_1}{\lambda_1 - \lambda_2}(d_1 - d_2) - d_3 + d_1 + e}. \quad . \quad (18)$$

which involves besides e only differences of distances, and the ratio $(\lambda_3 - \lambda_1)/(\lambda_1 - \lambda_2)$, which can be calculated without any knowledge of C from the observations of potential, and for these of course the properly corrected deflections may be taken.

Boltzmann found that no sensible difference in the values of K for ebonite, paraffin, sulphur, and rosin, was produced in the values of K by varying the time of charging or the amount of the charging battery. He also in one set of experiments tried the effect of excluding air from between the discs and the coatings of the condenser, by laying the discs on a mercury surface, and pouring a thin coating of mercury on a portion of the upper surface surrounded by an edging of paper.

Sp. Ind. Cap. of Ebonite, &c.

The results are given in the following table, in which the main columns I., II., III. give the results of experiments made with different distances between the plates. The first of the two sub-columns in each case gives the result for air between the disc and armatures, the second the result for mercury armatures.

Substance.	Values of K.						
	I.		II.		III.		Mean.
Ebonite . .	3·17	3·07	3·11	3·10	3·20	3·24	3·15
Paraffin . .	2·28	2·30	2·34	2·33	2·31		2·32
Sulphur . .	3·85		3·83				3·84
Rosin . . .	2·57		2·53				2·55

2. Method by Suspended Ball.

Boltzmann also determined the specific inductive capacities of the same substances by comparing the force on a small ball of the dielectric placed in a field of electric force of known intensity with the force on a conducting ball of equal size placed in the same field. This he did by hanging the ball as shown at s in Fig. 99, by a double thread from one end of a light rod, itself hung by a bifilar and forming therefore an arrangement akin to a torsion balance. The other end of the rod carried a mirror M by which the deflection of the balance could be obtained by means of a telescope and scale. The field was produced by a larger ball which was kept charged by means of a Leyden jar connected to its supporting rod.

Experiments were made for electrifications of the large ball of different durations,—(*a*) for a constant electrification of considerable duration, (*b*) for a comparatively short electrification, (*c*) for a rapidly alternating positive and negative electrification. The electrification (*b*) was obtained by making the charging and discharging contacts by the pendulum of a metronome, the electrification (*c*) by means of a vibrating tuning fork, one

2. Method by Suspended Ball.

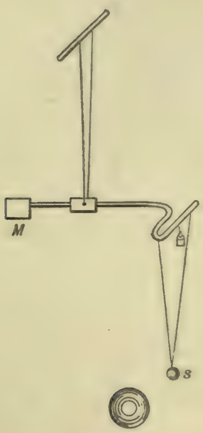


Fig. 99.

prong of which connected B alternately to each of two Leyden jars oppositely charged. By the result of p. 129 above, if we put $K_1 = 1$, and write K for K_2, and r denote the ratio of the force on the dielectric sphere to that on the conducting sphere, we have

$$\frac{K-1}{K+1} = r,$$

or

$$K = \frac{2r + 1}{1 - r} \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

For a sulphur ball, as will be seen from the table below, the force was practically the same for an alternating electrification of about $\frac{1}{200}$ sec. duration as for a long-continued electrification. Hence in this short interval the polarization of the dielectric was fully set up.

Sp. Ind. Cap. of Ebonite, &c.

The following are some of the results obtained, with the duration of electrification noted. For reference the mean value obtained with the condenser is added.

Substance.	K.			Value of K by Condenser.
	$\frac{1}{360}$ sec. to $\frac{1}{64}$ sec.	45 secs.	90 secs.	
Ebonite . . .	3·48	3·74	...	3·15
Paraffin . . .	2·32	8·12	...	2·32
Sulphur . . .	3·90	3·70	...	3·84
Rosin	2·48	5·28	5·61	2·55

The effect of increasing the duration of charge is therefore apparently to increase the specific inductive capacity, but in the cases of sulphur and ebonite to a much smaller extent than for the other two substances.

Sp. Ind. Cap. in different directions in Crystals.

Suspending a ball of crystallized sulphur with different diameters successively in the direction of the force of the field, Boltzmann found that the specific inductive capacity had different values in different

directions. For the greatest mean and least axes he found the following values:—

Sp. Ind. Cap. of different directions in Crystals.

	Greatest Axis.	Mean Axis.	Least Axis.
K	4·773	3·970	3·811

Experiments have been made by this method under Boltzmann's direction by Messrs. Romich and Nowak.* Results were obtained for (a) permanent electrification, and (β) for electrification reversed 64 times per minute. The values of K are given in the following table:—

	K	
	β	a
Glass	7·5	159
Fluorspar	6·7	7·1
Quartz	4·6	>1000
Calc Spar, perp. to axis . .	7·7	9·9
" parallel to axis . .	7·5	8·5
Selenium, freshly melted . .	10·2	151
Sulphur, mixed with Graphite	4	4·4

The difference between the results for permanent and for short continued electrification seem surprisingly great in some cases.

Klemencie has quite recently experimented on the specific inductive capacity of mica, and found it independent of the potential to which the condenser in

Klemencie's Results on Mica.

* *Wien. Ber.* 70 (1874). See also Wiedemann, *Lehre von d r Electricität*, Bd. ii. p. 34.

which the substance formed the dielectric, and practically independent of the duration of charge. K for the specimens used was 6·64.* So long as the condenser was kept thoroughly dry, the mica was found to insulate well and give constant results.†

Ayrton and Perry's Experiments on Ice.

By freezing distilled water in a shallow copper vessel in which was supported on three insulating feet a horizontal plate of copper in contact with the water surface, Professors Ayrton and Perry‡ made a condenser with ice as the dielectric. They then determined the capacity of this condenser and found from its dimensions the specific inductive capacity of ice. At $-13{\cdot}5°$ C. the value of K thus obtained was 22·168. It is of course to be remembered that the insulating power of ice is comparatively slight. Professors Ayrton and Perry found 2240×10^6 ohms for its specific resistance at $-12{\cdot}4°$ C.

Gordon's Experiments by Five-Plate Balance.

An extended series of experiments on solids has been made by Mr. J. E. H. Gordon,§ using a form of induction balance the idea of which is due to Sir William Thomson and Prof. Clerk-Maxwell. It is represented diagrammatically in Fig. 100. A, B, C, D, E are five parallel coaxial discs separated by intervals about an inch wide, of which the three A, C, E are six inches in diameter and the two B, D four inches in diameter. A and E are connected by a wire, the middle plate

* The value of K for mica is given as 5 in Jenkin's *Electricity and Magnetism*, but it is not stated on what authority.

† *Beiblätter*, vol. xii. No. 1. 1888.

‡ *Phil. Mag.* 1878, p. 43.

§ *Phil. Trans.* 1879, p. 417.

E is connected to the needle of a quadrant electrometer, the plates B, D to the electrodes of the pairs of quadrants. It is evident that, if a difference of potentials between C and A, E be established, it is

Gordon's Experiments by Five-Plate Balance.

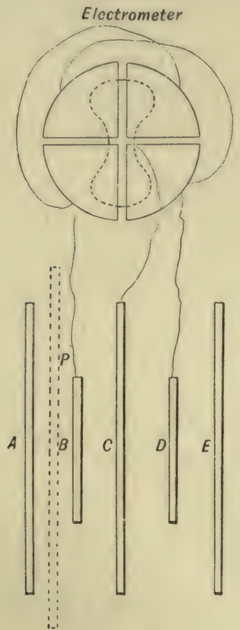


FIG. 100.

possible so to place A, B that the needle will not be affected, and it is also evident that when this position has been attained, the equilibrium will subsist whatever be the difference of potentials. The position of the plate A was adjustable by a micrometer screw, and

Gordon's Experiments by Five-Plate Balance.

equilibrium was attained by this means. It is to be noted that the effects of the edges of the plates are neglected.

The method of proceeding was therefore simply as follows. Having obtained equilibrium with air only between the plates, the experimenter introduced a plate P of the dielectric to be experimented on, and measured by means of S the distance through which A had to be displaced in order to restore equilibrium. This distance gave the thickness of a plate of air, equivalent to the plate P of the dielectric. The ratio of this thickness to the thickness of P is the specific inductive capacity of the material.

In the experiments the plates A, B, and C were connected to the terminals of an induction coil, the primary circuit of which was broken as many as 12,000 times a second by an interruptor arranged for the purpose. Thus the potential was rendered alternately positive and negative 12,000 times a second and all effects of absorption were obviated. It is to be noticed that here, as in some other experiments detailed above, the metallic plates were not in contact with the dielectric plate, and thus any passing over of electricity to the dielectric itself was avoided. The following are some of the results obtained :—

	K.		K.
Ebonite . .	2·284	Glass, Double-extra Dense Flint	3·164
Gutta Percha	2·462	„ Extra Dense Flint . . .	3·054
Sulphur . .	2·58	„ Light Flint	3·013
Shellac . . .	2·74	„ Hard Crown	3·108
Paraffin . . .	1·99	„ Common	3·243

Mr. Gordon found also by this method an apparent slow change in the specific inductive capacity of glass with lapse of time, a result which is to a certain extent corroborated by some preliminary experiments by Mr. T. Gray on the specific inductive capacity of glass soon after it had been heated to a high temperature;* but in view of the inaccuracy caused by the assumption that the plates of the balance may be taken as infinitely great, this slow change cannot be held to be proved. It seems probable that any slow change of specific inductive capacity, such as might correspond to the slow molecular change which goes on in glass which has been maintained for a long time at a nearly constant low temperature, and produces alteration of the zero point of a thermometer, would be of so small amount as to be imperceptible by any method of measurement yet devised.

Gordon's Experiments by Five-Plate Balance.

The values of K for glass obtained by Mr. Gordon are not in agreement with some previously obtained by Dr. John Hopkinson,† who experimented according to the method of comparison of capacities described above, p. 432. The capacity of a guard-ring condenser was compared with that of a sliding condenser (the identical instrument used in Gibson and Barclay's experiments described above) (1) when air only was the dielectric, (2) when a plate of glass was introduced between the plates. The guard-ring condenser is shown in Fig. 101, half in section half in elevation. k is the protected disc 15 centimetres in diameter with a gap 1 millimetre in breadth between it and the guard-ring, e e the opposite

Hopkinson's Earlier Experiments.

* *Phil. Mag.* Oct. 1880. † *Trans. R.S.* 1878, p. 17.

Guard-ring Condenser.

plate, h the guard-ring bearing a brass cylindrical box (not shown in the drawing) which forms a shield for the back of the protected disc. The guard-ring is insulated on a stiff frame of iron formed by two triangular pieces of iron a b, c d connected by three wrought-iron stays. The insulators are three ebonite legs g g, which are screwed to the tops of the stays. The attracting disc is carried on a screwed stem of 1/25 inch step, and can be raised or lowered without

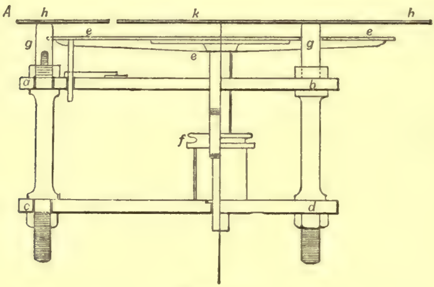


FIG. 101.

NOTE.—The protecting cylindrical box on the guard-ring is here omitted.

rotation by a nut f divided as a micrometer. Fig. 102 is a plan of the instrument with the brass backing removed. It shows the protected disc and its supports, which are two bars ll, ll of vulcanite attached to the back of the disc and resting on the upper surface of the guard-ring.

This instrument served also to measure the thickness of the glass plates used in the experiments. The screw f was turned until the brass plates were in contact, and

the micrometer reading taken; then the glass plate was placed above *e e*, which was screwed up until the plate came into contact with *h*, *k*, *h*. Slips of tissue paper were interposed between the ebonite legs *g g* and the plate *h h*, and the contact was judged by these slips becoming loose. A reading of the screw micrometer was taken for each slip, and the mean of the three Guard-ring Condenser.

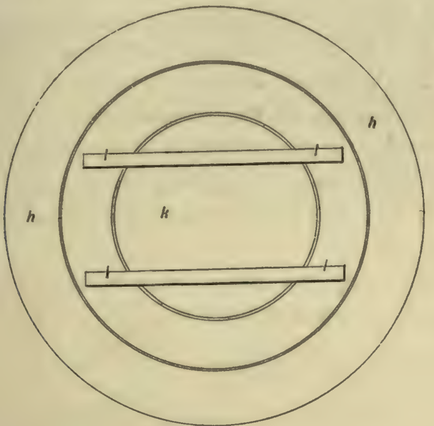


FIG. 102.

taken as the reading of contact. A correction was determined for the effect of bending of the plates and compression of the slips before their release.

A special switch supported above the guard-ring condenser enabled the connections to be made in the required order. A battery, in some cases of 48 in others of 72 small Daniell's cells, had its middle point connected to earth, one of its poles to *h k h*, and the

Guard-ring Condenser.

other to the inner coating of the sliding condenser, while the outer coating and the plate *e e* were connected to the electrometer case. Thus the inner plates of the two condensers were charged to equal and opposite potentials. Then one pair of quadrants of a Thomson's electrometer, both pairs of quadrants of which were connected to earth, were insulated, the guard-ring was connected to earth, and the protected plate and the insulated plate of the sliding condenser connected together and to the insulated quadrants of the electrometer. The direction of the electrometer deflection, if any, at the instant of the combination of the charges, was observed. If no deflection took place the guard-ring condenser and the sliding condenser had equal capacities, and the latter was adjusted until this was the case.

The following are mean results of two or more experiments for each substance:—

	Density.	*K.*
Glass, light flint . . .	3·2	6·85
,, double extra dense	4·5	10·1
,, dense flint . . .	3·66	7·4
,, very light flint .	2·87	6·57

The plates of glass were in most cases in contact with both plates of the condenser.

Hopkinson has since continued his investigations, and considered carefully the possible causes of the discrepance between his results and those obtained by Mr.

Gordon.* The result seems to leave no doubt that the five-plate-balance method with the sizes of plates and distances between them, used in Gordon's experiments, cannot give accurate results. The following conclusions among others were arrived at:—

Hopkinson's Criticism of Five-Plate Balance Method.

1. That the specific inductive capacity of glass is the same for $\frac{1}{1000}$ second, $\frac{1}{20000}$ second, or $\frac{1}{4}$ second discharge.

2. That it is independent of the potential to which the condenser is charged.

3. That the five-plate-balance is unreliable with the sizes of plates and distances apart used by Mr. Gordon. (A plate of brass between A and B, with an air-space of from eight to thirty-two millimetres, gave specific inductive capacity less than unity, instead of infinity. Different distances of the plates gave different values for glass.)

Hopkinson's later Experiments.

Hopkinson at the same time extended his former results, and applied his method of experimenting to the investigation of the specific inductive capacity of liquids. A flask of flint glass, with thin walls and a long thick neck, was filled up to the junction of the neck with strong sulphuric acid. A wire passing down through the neck connected the acid with a metal piece A (Fig. 103), supported on an insulating stand of ebonite. On this metal piece rested the horizontal arm of a kind of bell-crank (or L-shaped piece of metal pivoted at the angle). The flask was first charged by means of a battery and the potential measured by a quadrant

* Electrostatic Capacity of Glass (II.) and of Liquids, *Phil. Trans.* vol. 172 (1881), p. 372.

Hopkinson's later Experiments.

electrometer which was then detached and discharged. Then a previously deflected metallic pendulum, D, connected to earth through its supports, was released, and striking the vertical arm of the lever, connected the flask for an instant to earth and discharged it. The electrometer was then applied to

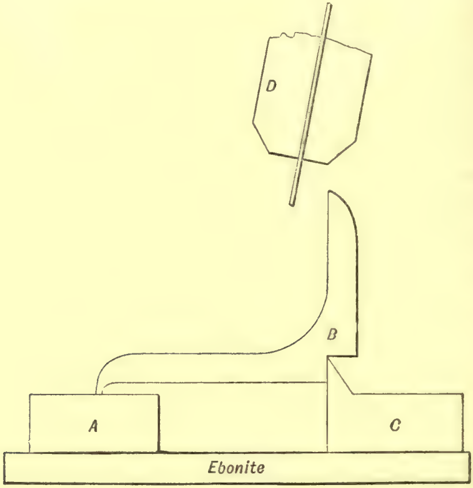


FIG. 103.

detect any residual charge. The leakage method described above, p. 403, was used to measure the duration of discharge. A paraffin condenser of known capacity had its plates connected for the time of discharge to be measured, first by a resistance of 256 ohms, then by a resistance of 512 ohms, and the remaining potential in each case was observed. These operations obviously

gave data for the calculation of the time interval t by (66) of Chap. VI.* With a duration of discharge of about $\frac{1}{2000}$ second, less than 3 per cent. of the original charge given by a battery of 20 elements remained. Longer and shorter times of discharge gave similar results. The practical result of all the experiments was that determinations of specific inductive capacity by observations of discharge may be taken as correct for glass if the period of discharge be anything between $\frac{1}{20000}$ sec. and $\frac{1}{2}$ sec.

Hopkinson's later Experiments.

The method adopted for determining the specific inductive capacity of glass plates was practically the same as that already described at p. 432. The guard ring and protected disc were first connected to one pole of a well insulated battery of 1,000 chloride of silver cells, the other pole of which was connected to the insulated plate of a cylindrical sliding condenser. Thus the two condensers were charged to equal and opposite potentials. By means of a special commutator changes of connections similar to those described above were made so as to combine the charges of the condensers, with the addition that the electrometer quadrants connected to the condensers after combination were immediately after insulated to avoid effects of residual charge. The capacity of the sliding condenser was adjusted till no electrometer deflection was produced.

The glass plate was then placed between the plates of the guard-ring condenser and the operations repeated until equilibrium was again obtained. The two results

* This mode of measuring a small interval of time is due to the late Mr. R. Sabine (*Phil. Mag.* 1876, 1st half year, p. 337).

Hopkinson's later Experiments.

gave the ratio of the capacities, and from the distance between the plates of the condenser and the thickness of the glass plate the value of K was found.

The capacity of the glass flask described above was determined in a similar way by aid of the sliding condenser, with a charging battery varying from 10 to 1,800 chloride of silver cells, with only a little over $\frac{1}{4}$ per cent. of alteration.

The values of K are given in the following table with the thicknesses of the plates, and for comparison the earlier results obtained by the same experimenter.

Hopkinson's results. Sp. Ind. Cap. of Glass, &c.

Substance.	Density.	Thickness of Plate in mms.	K.	Value of K formerly obtained.
Glass, Double - extra Dense Flint .	4·5	4·5	9·896	10·1
,, Dense Flint .	3·66	16 57	7·376	7·4
,, Light Flint . .	3·2	15·04	6·72	6·83
,, ,, ,, . .	—	10·75	6·69	6·85
,, Very Light Flint	2·87	12·70	6 61	6·57
,, Hard Crown .	2 485	14·62	6 96	—
,, Plate	—	6·52	8·45	—
Paraffin	—	20·19	2·29	—

Sp. Ind. Cap. of Liquids. Hopkinson's Experiments.

Dr. Hopkinson obtained results also for liquids by the method just described.* The space between two co-axial metal cylinders was filled with the liquid to be experimented on. These two cylinders connected together formed one coating of a condenser of which

* *Phil. Trans. loc. cit.*

the liquid formed the dielectric, and the other coating was given by a cylinder suspended from an ebonite plate above, and immersed in the liquid. The latter plate was charged and the other connected to earth, and the capacity compared with that of the oppositely charged sliding condenser. The capacity of the same apparatus with air as the dielectric had previously been obtained in the same way, and the results gave at once the value of K for the liquid. The following table gives some of the results. The column headed μ^2_∞ contains for the purpose of comparison the square of the index of refraction of the liquid for light of infinite wave length. This was calculated from the formula $\mu_\infty = A + B/\lambda^2$ from observations of the index of refraction which were made on each of the substances for the Fraunhofer rays C, D, F, G, of the spectrum.

Sp. Ind. Cap. of Liquids. Hopkinson's Experiments.

Name of Liquid.	K.	$\mu^2\ \infty$
Petroleum Spirit	1·922	1·92
Petroleum Oil, Field's	2·07	2·075
" " Common . . .	2·10	2·078
Ozokerite	2·13	2·086
Turpentine, Commercial . . .	2·23	2·128
Castor Oil	4·78	2·153
Sperm Oil.	3·02	2·135
Olive Oil	3·16	2·131
Neat's Foot Oil	3·07	2·125

The closeness of the agreement between the numbers for K and for μ^2_∞ for the mineral oils and for turpentine is very remarkable. The divergence in the other cases

is to be expected, as from the composition of the substances it is probable that the results included effects of electrolytic action.

Silów's Experiments. First Method.

Results with which Hopkinson's agree very well had been previously obtained for turpentine, benzene, and petroleum by Silow.* Two series of experiments were made. In the first a very ingenious and simple method was employed. A kind of quadrant electrometer was constructed by pasting on the inside of a cylindrical glass vessel, 10 centimetres deep and 15 centimetres in diameter, four symmetrically placed vertical strips of tinfoil each 10 centimetres broad, and joining the opposite pieces together by strips across the bottom. Within was hung a platinum needle of the shape of an inverted T, in which the vertical pieces at the ends of the horizontal cross-piece were semi-cylinders of platinum. The needle was left uncharged, and one of the pairs of strips was connected to earth and the other charged to a convenient potential. The deflections of the needle for the same difference of potential (1) with the vessel filled with air, (2) with the liquid under experiment, were observed, and it was assumed that the angles of deflection were proportional to the specific inductive capacities in the two cases. This would have been strictly true of the angles through which a torsion head at the top of the suspension thread would have had to be turned if the needle had been brought back in both cases to a position of equilibrium after deflection.

* *Pogg. Ann.* 156 (1875), p. 389, and Wiedemann, *Die Lehre von der Elektricität,* Bd. ii. p. 45.

For two kinds of turpentine, I., II., and for petroleum he obtained :—

Silow's Results by First Method.

	K.	$\mu^2 \infty$
Turpentine I., mean of three experiments	2·173	2·129
Turpentine II.	2·221	
Petroleum	2·037	2·148

Silow's Experiments. Second Method.

A second set of experiments was made by Silow by a method similar to that described above, p. 448. A condenser formed of two gilded circular plates kept $1\frac{1}{2}$ mm. apart by small pieces of ebonite, and enclosed within a glass vessel covered on its interior surface with tinfoil, had one of its plates alternately connected to earth and to one pole of a water battery of 175 zinc-copper elements. The connections were made by a rotating commutator kept running at a constant speed sufficiently great to give a constant deflection of the needle of a galvanometer placed in the charging or discharging circuit. Three deflections were taken (1) with the vessel filled with air, (2) with the liquid under experiment in the vessel and therefore between the plates, (3) with only the joining wires attached. Denoting by a, β, γ, these deflections corrected so as to be proportional to the currents, we have for the ratio of the capacity of the apparatus with the liquid between its plates, to its capacity with air between the plates $(\beta - \gamma)/(a - \gamma)$, that is for the liquid

$$K = \frac{\beta - \gamma}{a - \gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Silow's Results by Second Method.

Different battery powers applied gave the same values for K. The following are the mean values of K for the substances mentioned, with the values of $\mu^2{}_\infty$ for comparison.

Substance.	K.	$\mu^2{}_\infty$
Turpentine	2·153	2·134
Benzene	2·198	2·196
Petroleum, first specimen . . .	2·071	2·048
Petroleum, second specimen . .	2·037	2·048

Quincke's Experiments.

Some interesting experiments on the specific inductive capacity of liquids have also been made by Quincke.* According to the theory of Faraday and Maxwell, referred to at p. 133 above, there is, at every point of the electric medium, a tension along the lines of force, and an equal pressure at right angles to that direction, the amount of which reckoned in units of force per unit of area is $KF^2/8\pi$ where F is the resultant electric force at the point. Quincke's method amounted to measuring not only the tension, but the pressure also, in different liquid dielectrics, and his results besides giving (1) from the observed tension, (2) from the pressure, values of K which he compared with those obtained by the ordinary condenser method, are interesting in their bearing on electrical theory.

Measurement of Tension along Lines of Force.

His apparatus for the measurement of the tension consisted of two horizontal circular plates placed a short distance apart in a glass vessel. The upper plate

* Wied. *Ann.* 19 (1883).

was suspended from one end of the beam of a balance, and was connected to earth. The lower plate was charged by means of a battery of Leyden jars, the outer coatings of which were to earth. The potential was observed in arbitrary units by means of a Thomson's standard electrometer (see p. 281 above). The attraction of the upper plate towards the lower was then measured by weights put on the other scale of the balance. The mean pull per unit of area was therefore obtained.

Measurement of Tension along Lines of Force.

Now from what has been proved above (pp. 111, 136) it follows that the force f, per unit of area on any part of the upper plate not near the edge is $2\pi\sigma^2/K$, and we have $\sigma = -KF/4\pi = -KV/4\pi d$ if V be the difference of potentials, d the distance between the plates. Hence

$$f = \frac{KV^2}{8\pi d^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The weighing therefore gave, taking the mean pull as nearly enough equal to f, directly the tension.

By comparison of results for two different media using the same value of V for both cases, the ratio of the values of K could be at once obtained. Thus if f_1, f_2, be the tensions, and the corresponding specific inductive capacities determined in this manner be denoted by K_{f_1}, K_{f_2}, we have

Deduction of Sp. Ind. Cap.

$$\frac{K_{f_1}}{K_{f_2}} = \frac{f_1}{f_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

The pressure at right angles to the lines of force was found in an ingenious manner. The upper disc of the apparatus just described was removed and replaced by

Measurements of Pressure across Lines of Force.

Measurements of Pressure across Lines of Force.

a plate of the same diameter with a short vertical tube at its centre, by means of which communication could be obtained with the space between the plates. Attached to this vertical tube was an india-rubber bag which could be cut off by means of a stopcock. A branch tube communicated with an ordinary open U manometer containing bisulphide of carbon. Enough of air was blown by the rubber bag into the space between the plates to form a flat bubble of from 2 to 5 centimetres in horizontal diameter, bounded by the plates above and below. The stopcock was closed and the pressure was read off on the manometer. The lower plate was now charged to the same potential as before while the upper plate was connected to earth. The increase of pressure was read off from the manometer, and gave the difference of pressures in the air and the liquid due to the electrification.

Deduction of Sp. Ind. Cap.

If h be the difference of heights of the liquid produced by the electrification, and ρ the density of the liquid, we have, denoting the value of K determined in this way by K_p, and the acceleration due to gravity by g

$$gh\rho = \frac{K_p - 1}{8\pi} \frac{V^2}{d^2} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

if K be taken $= 1$ for air.

Using the value of f given in (21) for the same medium, this gives

$$K_p = \frac{gh\rho}{f} K_f + 1 \quad . \quad . \quad . \quad . \quad . \quad (24)$$

The following are some of the results obtained.

Quincke's Results.

Substance.	Density at Temp. stated.		Temp. of Exp.	Value of K Sp. Ind. Cap.		
				By Condenser K.	By Tension K_f.	By Pressure K_p.
		°C.	°C.			
Sulphuric Ether	·7205	14·9	6·60	3·364	4 851	4·672
Bisulphide of Carbon I.	1·2760	12·20	7·50	2·217	2·669	2·743
" " " II.	1·2796	10·20	12·98	1·970	2·692	2·752
Benzene (from Coal Tar)	·8825	15·91	13·20	1·928	2·389	2·370
" (from Benzoic Acid)	·8822	17·64	14·40	2·050	2·325	2·375
Light Benzene	·7994	17·20	11·60	1·775	2·155	2·172
Colza Oil	·9159	16·40	16·41	2·443	2·385	3·296
Turpentine	·8645	17·10	16·71	1·940	2·259	2·356
Petroleum	·8028	17·00	16·62	1·705	2·138	2·149
Ether 5 vols. + 1 vol. Bisulph. of Carbon	·8134	16·40	8·50	2·871	4·136	4 392
Ether 1 vol. + 1 vol. Bisulph. of Carbon	·9966	16·60	10·50	2·458	3·539	3·392
Ether 1 vol. + 3 vols. Bisulph. of Carbon	1·1360	17·40	5·30	2·396	3·132	3·061

The values of K obtained by tension and pressure here seem uniformly greater than those obtained by the condenser method, which must be regarded of course as the true values. But they agree very well with one another, and go far to prove the equality of the pressure and tension.

Quincke's Results corrected for Connections.

It was pointed out by Dr. Hopkinson that * perhaps the capacity of the key and connecting wires might be appreciable, and that if so the values of K given for the condenser method in the above table would be increased by the correction. This was found by Professor Quincke to be the case, and the following corrected results obtained by him are given by him in a note to Dr. Hopkinson's paper.

	Values of Sp. Ind. Cap.	
	By Condenser K.	By Tension K_f.
Sulphuric Ether . . .	4·211	4·394
Bisulphide of Carbon .	2·508	2·623
" " .	2·640	2·541
Benzene	2·359	2·360
Petroleum	2·025	2·073

This shows that for these substances K, K_f, K_p are sensibly equal. Further the experiments seem to confirm fairly well the theoretical values $KF^2/8\pi$ for the pressure within the medium.† (See also p. 491, below).

* *Proc. R. S.* vol. xli. 1886.

† The whole question of the stress in the medium requires further

Dr. Hopkinson has more recently * made experiments on the specific inductive capacity of a number of oils and other liquids. The method adopted was a modification of the five-plate balance method described above. The arrangement of apparatus is shown in Fig. 104. Two air condensers *E*, *F*, of determinate and nearly equal capacity, and two adjustable sliding condensers *I*,

Hopkinson's Modification of Five-Plate Balance Method.

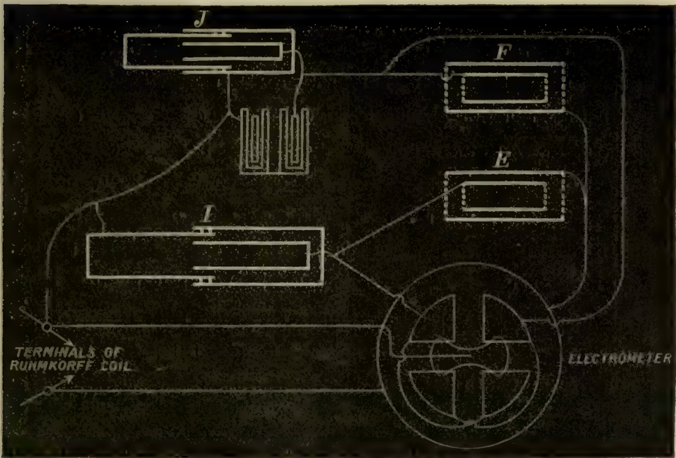


Fig. 104.

experimental investigation. Valuable results as to the state of strain in transparent, solid, and liquid media have however been furnished by the experiments of Kerr and others on Double Refraction produced by electrification. The experiments of Quincke on Change of Volume produced in dielectrics by the same cause are also of great importance in the same connection. These are however ˙ easurements we cannot here enter into.

* *Proc. R. S.* Oct. 1887.

Hopkinson's Modification of Five-Plate Balance Method.

J, were joined as shown like the four branches of a Wheatstone bridge. The inner coatings of E, I were joined to one pair of quadrants of an electrometer, and those of F, J to the other pair of quadrants. To the inner coating of J could be attached the inner plate of a liquid condenser containing the substance to be experimented on. The outer coatings of E, F were connected to the case of the electrometer and to one terminal of an induction coil; the outer coatings of I, J were connected to the needle of the electrometer and to the other terminal of the induction coil.

In order that there might be no deflection of the electrometer needle it was necessary that the capacities of E and I should be in the same ratio as those of F and J respectively. An adjustment of one or both of the sliding condensers was made until this relation was fulfilled in each of four cases, (1) when no fluid condenser was introduced, (2) when the condenser without the interior plate, but fitted with a "dummy" to represent the necessary supports or connexions outside the liquid, was connected to J, (3) when the complete condenser charged with air was added to J, (4) when the complete condenser charged with liquid was connected to J. Assuming for simplicity the sliding condenser I to remain unaltered, and x, y, z, z_1 to be the respective readings of J in the four cases, we must have

$$\frac{\text{Capacity of condenser with liquid}}{\text{Capacity of same condenser with air}} = K = \frac{x - z_1 - (x - y)}{x - z - (x - y)}$$

$$= \frac{y - z_1}{y - z} \quad . \quad . \quad . \quad . \quad . \quad . \quad (257)$$

The following is an abstract of the results obtained:

Hopkinson's Results for Oils, &c.

	K.	μ^2_D for line.
Colza Oil, six samples . .	3·07 to 3·14	
„ „ another sample *	3·23	
Arachide	3·17	
Sesame	3·17	
Linseed Oil, raw	3·37	
Castor Oil	4·82	
„ „ another sample .	4·84	
Ether	4·75	
Carbon Bisulphide . . .	2·67	
Amylene	2·05	1·9044

It is to be noted with respect to colza oil that, as given by Quincke (p. 483 above), the value of K_p is 3·296 and of K_f 2·385.

Dr. Hopkinson also experimented with the following liquids of the benzene series, for which also he determined the index of refraction μ_D for the line D of the spectrum.

	K.	μ^2_D.
Benzene	2·38	2·2614
Toluene	2·42	2·2470
Xylene.	2·39	2·2238
Cymene	2·25	2·2254

Application of Method to Solids.

The same method, but with a guard-ring condenser instead of the fluid condenser as shown in Fig. 105, was applied to the measurement of the specific inductive

* Doubtful as to purity.

Application of Method to Solids.

capacity of solids. The connections shown in Fig. 105 were first made, that is the guard-ring and protected disc both connected to the inner coating of J. The arrangement was then adjusted to balance, then the guard-ring remaining connected to J, the protected disc was transferred to I and balance again obtained. The difference of the readings of the sliding condenser

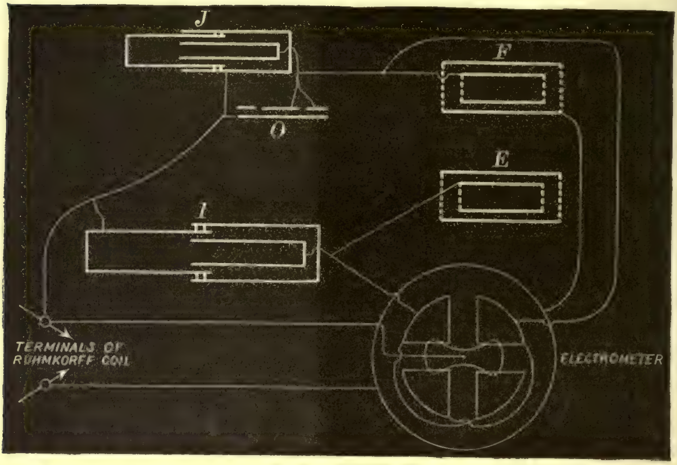


Fig. 105.

gave on an arbitrary scale the capacity of the guard-ring condenser for the given distance of the plates apart. These operations were then repeated with a plate of the substance for which K was to be found placed between the plates of the guard-ring condenser.

Only three substances were experimented on, with

the following results. The previously obtained values (p. 476 above) are given for the first two for comparison.

Application of Method to Solids.

	K.	Previously found value of K.
Flint Glass, double extra dense	9·5	9·896
Paraffin Wax	2·31	2·29

Rock salt was the third substance with a result of 18 for K, but the sample was very rough and too small, and possibly conducted so greatly as to affect the result. In these experiments the effect of the connecting wire of the guard-ring condenser was not allowed for.

Negreano's Experiments on Hydro-Carbons.

Negreano * has applied the five-plate balance method to the determination of the specific inductive capacity of a number of hydrocarbons of homologous chemical composition. The balance was arranged with its plates horizontal and well insulated on ebonite rods; the diameter of the larger plates was 16 centimetres, of the smaller 12 centimetres, and the distance of adjacent plates apart 1 centimetre. The liquid experimented on was placed on a flat shallow dish attached to the ebonite supports between the uppermost plate and that next to it. Balance was obtained (1) with the instrument used simply as an air condenser, (2) with the empty dish in position, (3) with the liquid in the dish. The corresponding positions of the movable plate were obtained by a micrometer. Another micrometer measured the thickness of the stratum of liquid. The

* *Comptes Rendus*, tome civ. 1887.

Negreano's Experiments on Hydro-Carbons.

index of refraction μ_D was also determined for the D line in the case of each liquid.

It was found that the value of K increases as the composition of the substance becomes more complicated, and that the value of $(K-1)/(K+2)\rho$ where ρ is the density is approximately constant. The following is a synopsis of the results :—

	Temp.	Density.	K.	μ_D.
Benzene, C_6H_6, with thiophene	26°	·8803	2·3206	1·4974
,, ,, ,, ,, another specimen	25	·8756	2·2988	1·4978
,, ,, pure . . .	14	·8853	2·2921	1·5062
Toluene, C_7H_8	27	·8608	2·242	1·4912
,, ,, ...	14	·8711	2·3013	1·4984
Xylene, C_8H_{10}	27	·8554	2·2679	1·4897
Metaxylene, C_8H_{10}	12	·8072	2·3781	1·4977
Pseudocumene, C_9H_{12} . . .	14	·857	2·4310	1·4837
Cymene, $C_{10}H_{14}$	19	·851	2·4706	1·4837
Terebenthene, $C_{10}H_{16}$ . . .	20	·875	2 2618	1·4726

It will be noticed that the value of $\sqrt{K}$ is only a little greater than μ_D in each case, and that $(K-1)/(K+2)\rho$ has the value ·34 approximately in the first six cases and the last, and is slightly greater in the remaining three.

Cohn and Arons' Experiments in Liquids.

Experiments on liquids have also been made by E. Cohn and L. Arons. Two quadrant electrometers were employed, one with air filling the quadrants, the other specially designed to contain the liquid experimented on as in Silow's method described above, p. 478. One pair of quadrants of each electrometer was con-

Cohn and Arons' Experiments in Liquids.

nected to one terminal of a Helmholtz induction coil, the other pair of quadrants, the needle and the case were connected to earth and to the other terminal of the coil. Denoting by δ_1, δ_2 the (corrected) deflections on the ordinary and special electrometers respectively when both are filled with air, δ'_1, δ'_2 the corresponding deflections when the special electrometer contains the liquid, we get easily by (20) above

$$K = \frac{\delta_2 \delta'_1}{\delta_1 \delta'_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Results.

The following results were obtained :—

	K.
Distilled Water	76
Ethyl Alcohol	26·5
Amyl Alcohol	15
Petroleum	2·04
Xylene, two kinds . . .	2·39
	2·36

The numbers here given it will be observed are high in the first three cases. These substances have however considerable conductivity, which would tend of course to give an apparently high specific inductive capacity. The authors believe that the results are correct within 5 per cent.

Prof. Quincke * has re-examined the question of the values of K for liquids obtained by the different methods, as described above. All liquids experimented on except

* *Wied. Ann.* 33, 1888.

Result. colza oil give practically the same result whatever the method employed. For that substance however the result stated above holds, that is the pressure method gives the highest value, the electrical balance the lowest, and the condenser method a mean value; and this anomaly was found to hold good for different kinds of colza. That it could not be due to electrolytic action was clear from the fact that the products of decomposition at the condenser plates could not alter the pressure at the surface of the bubble.

Prof. Quincke * also measured the index of refraction of pure ether for ultra-red rays by passing them through the medium and receiving them upon a thermopile. He found that for pure ether $K = 4{\cdot}3$, and that for ultra red rays its index of refraction is less than 2. The substance seems therefore not to conform to Maxwell's relation.

Sp. Ind. Cap. of Gases. Boltzmann's Experiments. Determinations of the specific inductive capacity of gases have been made by Boltzmann† and by Professors Ayrton and Perry.‡ Boltzmann's method was as follows. A condenser consisting of two horizontal circular plates was supported within a closed metallic vessel, through the walls of which passed wires to make connection with the plates, and which could be connected with an air-pump or a gas generating apparatus. Two metallic plates were placed above and two below the condenser to preserve it at a uniform temperature. The vessel was exhausted, then one plate of the condenser A was

* *Wied. Ann.* 32. No. 12. 1887.

† *Wien. Ber.* 69 (1874); *Pogg. Ann.* 15 (1875).

‡ *Trans. Asiatic Society of Japan* (1877).

Sp. Ind. Cap. of Gases. Boltzmann's Experiments.

charged by being connected to one terminal of a battery of 300 Daniell's cells, while the other plate B and the other terminal of the battery were connected to earth. B was then disconnected from earth and connected to the insulated electrode of an electrometer which had been previously brought to zero potential. The electrometer showed no deflection, proving that there was no leakage. The charge on A therefore remaining constant, it was found in accordance with theory that the admission of air altered only the specific inductive capacity between the plates, and therefore the potential of A, but not the potential of B which remained zero. After the admission of air the potential of A was restored to its original value, and the change of potential of B read off on the electrometer. The number of cells was then increased by one, and the increased potential of B again read off. The ratio of the specific inductive capacities could now be calculated.

If V_1, V_2 be the potentials of A before and after the admission of air, and K_1, K_2 the corresponding specific inductive capacities, we have $V_2/V_1 = K_1/K_2$. Hence by the restoration of the potential to V_1 the potential of B was increased by an amount proportional to $V_1 - V_2$, that is by an amount $m(1 - K_1/K_2)$ where m is a constant. By the increase of the number of cells from n to $n + 1$ the increase of the potential of B was therefore $mV_1(n + 1)/n$. Hence calling these changes as measured by the electrometer δ, δ', we have $\delta/\delta' = n(1 - K_1/K_2)\ (n + 1)$, or

$$K_2 = \frac{n\delta'}{n\delta' - (n + 1)\delta} K_1 \quad . \quad . \quad . \quad . \quad (27)$$

Sp. Ind. Cap. of Gases. Boltzmann's Experiments.

It was found by Boltzmann that the alteration of capacity was very nearly in simple proportion to the alteration of pressure of the air, and that the effect of alteration of temperature was only that corresponding to the consequent alteration of pressure. Hence if we denote by K the specific inductive capacity of air under pressure equal to that due to p millimetres of mercury under standard circumstances, suppose that for absolute vacuum to be unity, and assume the proportionality to hold for all pressures, we may write

$$K = 1 + \frac{kp}{760} \quad . \quad . \quad . \quad . \quad . \quad (28)$$

where $1 + k$ is the specific inductive capacity of air at standard atmospheric pressure.

By (27) and (28) putting p_1, p_2 for the pressures corresponding to K_1, K_2, we get

$$k = 760 \frac{n\delta' + (n+1)\delta}{n\delta'(p_2 - p_1) - (n+1)p_2\delta} \quad . \quad . \quad (29)$$

Boltzmann found similar results to hold for other gases than air, and gave the following values for K at standard atmospheric pressure. The value of $\sqrt{K}$ is given also for comparison with the index of refraction.

The Value of K.

Gas.	K.	$\sqrt{K}$.	μ.
Air	1·000590	1·000295	1·000294
Carbonic Acid	1·000946	1·000473	1·000449
Hydrogen	1·000264	1·000132	1·000138
Carbonic Oxide	1·000690	1·000345	1·000340
Nitrous Oxide	1·000994	1·000497	1·000503
Olefiant Gas	1·001312	1·000656	1·000678
Marsh Gas	1·000944	1·000472	1·000443

Ayrton and Perry's Experiments.

In Ayrton and Perry's method the capacities of two condensers were compared with different gases at different pressures between the plates of one of them, while the other had continually air at ordinary temperature and pressure for its dielectric. The latter condenser consisted of a square horizontal uninsulated plate of tin-foil of 1815 square centimetres area, cemented to the upper surface of a plate of hard wood which rested on the horizontal top of a block of stone, and an insulated upper plate of the same size supported on ebonite levelling screws, the lower ends of which rested on the stone. The other condenser was contained within an air-tight rectangular vessel of sheet brass, and consisted of eleven parallel plane plates, each 324 square centimetres in area, kept at equal distances of three millimetres apart in racks of ebonite. The first, third, &c., and last plates, reckoning from one side, were connected to the case, the other plates were insulated and connected to a platinum wire passing out through a glass tube 35½ centimetres long to the outside of the case. This glass tube, which had been chemically cleaned and covered with paraffin, to prevent leakage over the surface, was very carefully cemented into a brass socket attached to the metallic case, and was nowhere in contact with the platinum wire except at the outer end, where it was drawn to a point and hermetically sealed. Cement contained in a metal cap surrounding the junction of the tube and socket prevented leakage there, and a second cap filled with cement surrounded the point of the tube, and guarded the point from being broken by motion of the wire.

Ayrton and Perry's Experiments.

By means of another tube the case could be filled with the gas to be experimented on or connected to a Sprengel or other pump by which the required degree of exhaustion was produced. This tube was made of special form to prevent mercury from the Sprengel pump from passing by any accident into the condenser case.

The method of making a determination was as follows. The insulated plates of the condenser were charged to equal and opposite potentials in the following manner:—The battery of 87 Daniell's cells had its poles joined by a resistance of 10,000 ohms, and by means of a reversing key one terminal a of this coil was connected to the insulated plate of one condenser, while the other terminal b was connected to earth; then b was connected by the reversing key to the insulated plate of the other condenser and a to earth.

The battery was then removed and the charged plates connected together, and with the insulated electrode of a quadrant electrometer of which the other electrode and case were to earth, and the reading taken.

If the potential of each condenser was numerically V, the capacity of the constant air condenser C_1, and the capacity of the other C_2, the charge left after the two condensers were connected was $V(C_1 - C_2)$, supposing the constant condenser to have been positively charged. The corrected deflection α shown by the electrometer was therefore $mV(C_1 - C_2)/(C_1 + C_2)$ where m is a constant.

To eliminate m and V the terminals of the battery were kept joined by the resistance of 10,000 ohms, and

one terminal was connected to earth, while a point on the resistance was connected to the insulated quadrants of the electrometer now detached from the condensers. The difference of potentials of the battery between the extremities of the resistance was $2V$, and if the resistance intercepted between the terminals of the electrometer be denoted by R, the difference of potentials shown by the corrected deflection β of the electrometer was $2VR/10000$. We have therefore $\beta = 2mVR/10000$. Hence

Ayrton and Perry's Experiments.

$$\frac{a}{\beta} = \frac{10000}{R}\frac{C_1 - C_2}{C_1 + C_2} = \frac{10000}{R}\frac{1 - \dfrac{C_2}{C_1}}{1 + \dfrac{C_2}{C_1}}. \quad . \quad (30)$$

This enabled the ratio C_2/C_1 of the capacities to be calculated. Another experiment made with C_2 changed by alteration of the medium, gave at once the ratio of the two values of C_2, that is of the specific inductive capacities in the two cases.

The following table gives the mean results for many experiments in different gases at standard pressure: taking the value of K for air as unity.

Dielectric.	K.
Vacuum	·9985
Air	1·0000
Carbonic Acid	1·0008
Hydrogen	·9998
Coal Gas	1·0004
Sulphurous Acid	1·0037

Ayrton and Perry's Experiments.

It was observed that when air was allowed to mix with the carbonic acid the value of K more and more nearly approached unity.

Experiments on the specific inductive capacity of a high sprengel vacuum have been undertaken by a Committee of the British Association consisting of Professors Ayrton and Perry, Prof. O. J. Lodge, and Mr. J. E. H. Gordon. A preliminary report has been presented * containing a plan of experimenting and some results which seem to show that at a pressure of about $1/10^6$ of an atmosphere the specific inductive capacity is ·6 or ·8 per cent. less than that for ordinary air. The committee have not yet concluded their labours.

Effects of Change of Temperature.

The results of some recent experiments made by Mr. Cassie in the Cavendish Laboratory, on the effect of rise of temperature in increasing the specific inductive capacity of solid dielectrics, are quoted by Prof. J. J. Thomson in his work entitled *Applications of Dynamics to Physics and Chemistry*, p. 102. The coefficient of increase of specific inductive capacity per degree centigrade, that is, the value of $1/K. \, dK/d\theta$, is, for $\theta = 30°$, ·002 for glass, ·0004 for mica, and ·0007 for ebonite. From this Prof. J. J. Thomson has shown that if the electric displacement be f, there must at 30° be $·002 \times 30 \times 2\pi f^2/K$ dynamical units of heat supplied to unit of volume of glass to preserve its temperature constant when it is electrified. The corresponding quantities of heat for mica and ebonite are respectively

* *Brit. Assoc. Rep.* 1880.

$\cdot 0004 \times 30 \times 2\pi f^2/K$, $\cdot 0007 \times 30 \times 2\pi f^2/K$. But the electrical work done in charging is (p. 133 above) in each case $2\pi f^2/K$. Hence in the case of glass the heat thus absorbed during charging is about two-thirds of the work done in charging.

Effects of Change of Temperature.

NOTE.—An account of the determinations of specific inductive capacity made by Schiller by the method of elcetrical oscillations will be found in Volume II.

APPENDIX.

NOTE.

Recommendations of the Paris Congress and the British Association as to Practical Electrical Units.

At the meeting of the Electrical Congress held in Paris in 1884, it was decided to adopt for the present, as practical unit of resistance, a resistance equal to that of a uniform column of mercury one square millimetre in section, 106 centimetres in length, and throughout at the temperature 0° C. The mercury column thus specified expressed approximately and in round numbers the value of the ohm according to the latest and most accurate experiments. It was resolved to give this unit the name *Legal Ohm.*

The Congress also arrived at certain conclusions regarding the practical units of current, electromotive force, quantity of electricity, and electrostatic capacity, as follows :—

(1) That the unit of current should be called the *Ampere*, and be defined as $\frac{1}{10}$ of a C.G.S. electromagnetic unit of current.

(2) That the *Volt* or practical unit of electromotive force, or difference of potentials, should be defined as the electromotive force required to maintain a current of one ampere through a resistance of one ohm.

(3) That the unit quantity of electricity should be called the *Coulomb*, and be defined as equal to the quantity of electricity transferred by a current of one ampere in one second.

(4) That the *Farad* or practical unit of capacity should be the capacity of a conductor which is charged to a potential of one volt by one coulomb of electricity.

The British Association at its meeting in 1886 agreed that the Committee on Electrical Standards should recommend to Her Majesty's Government :—

(1) "To adopt for a term of ten years the Legal Ohm of the Paris Congress as a legalized standard sufficiently near to the absolute Ohm for commercial purposes.

(2) "That at the end of the ten years' period the Legal Ohm should be defined to a closer approximation to the absolute Ohm.

(3) "That the resolutions of the Paris Congress with respect to the Ampere, the Volt, the Coulomb, and the Farad, be adopted.

(4) "That the Resistance Standards belonging to the Committee of the British Association on Electrical Standards, now deposited at the Cavendish Laboratory at Cambridge, be accepted as the English Legal Standards conformable to the accepted definition of the Paris Congress."

A full account of the electromagnetic system of units and of the derivation of the various practical units will be given in Volume II., but as the ampere, volt, &c. have been referred to above, p. 415, the following sketch may be found here useful. This system of units is based on a definition of unit magnetic pole, or (which is the same thing) unit quantity of magnetism, precisely similar to that given on page 3 above for unit quantity of electricity. Unit magnetic pole is that pole which placed (in air) at unit distance from an equal pole of the same kind is repelled with unit force. When the fundamental units are the centimetre, the gramme, and the second, the unit distance and the unit force of this definition are respectively one centimetre and one dyne. Now by the discovery of Oersted, as explained by the theory of Ampère, a current of electricity produces magnetic force at every point of the surrounding space. The intensity of this field at any point is measured by the force which a unit magnetic pole would experience if placed at that point. Unit current is then that current which flowing in a thin circular conductor produces a magnetic field of $2\pi r$ units intensity, where r is the radius of the circle into which the conductor is bent, and π is the ratio of the circumference of a circle to its diameter. When r is one centimetre and the intensity of the field 2π dynes, the current is one C.G.S. electromagnetic unit in strength.

The definitions of the volt, &c. follow from that of unit current as stated above in the recommendations of the Paris Congress. The *Microfarad* (referred to at p. 406 above) is one-millionth of the Farad, and is a more convenient unit than the latter, which is so large as to give somewhat small fractional numerics for the capacities of ordinary condensers.

TABLE I.

CROSS-SECTION OF ROUND WIRES, WITH RESISTANCE, CONDUCTIVITY, AND WEIGHT OF HARD-DRAWN PURE COPPER WIRES, ACCORDING TO THE NEW STANDARD WIRE GAUGE LEGALISED BY ORDER IN COUNCIL, AUGUST 23, 1883.

Temperature 15° Cent.

Descriptive No.	Diameter.		Area of Cross-section.		Resistance.		Conductivity.		Weight. (Density=8·95)	
	Ins.	Cms.	Sq. Ins.	Sq. cms.	Legal Ohms per Yard.	Legal Ohms per Metre.	Yards per Legal Ohm.	Metres per Legal Ohm.	Lbs. per Yard.	Grms. per Metre.
0000000	·500	1·270	·1963	1·267	·000125	·000136	8055	7365	2·285	1134
000000	·464	1·179	·1690	1·091	·000144	·000157	6937	6343	1·970	976·3
00000	·432	1·097	·1466	·946	·000166	·000182	6013	5498	1·706	846·3
0000	·400	1·016	·1257	·811	·000194	·000213	5054	4714	1·463	725·6
000	·372	·945	·1087	·701	·000225	·000245	4459	4077	1·265	627·6
00	·348	·884	·0951	·614	·000256	·000280	3901	3568	1·107	549·6
0	·324	·823	·0824	·532	·000296	·000323	3384	3093	·960	476·1
1	·300	·762	·0707	·456	·000345	·000377	2899	2652	·823	408·1
2	·276	·701	·0598	·386	·000408	·000446	2454	2244	·696	345·4
3	·252	·640	·0499	·322	·000489	·000536	2046	1871	·581	288·0
4	·232	·589	·0423	·273	·000577	·000631	1734	1586	·492	244·1
5	·212	·538	·0353	·228	·000691	·000756	1451	1324	·411	203·8
6	·192	·488	·0290	·187	·000842	·000921	1197	1086	·337	166·8
7	·176	·447	·0243	·157	·00100	·00110	988	912	·283	140·5
8	·160	·406	·0201	·130	·00122	·00135	824	748	·234	116·1
9	·144	·366	·0163	·105	·00149	·00164	669	611	·190	94·0
10	·128	·325	·0129	·0830	·00190	·00208	528	482	·150	74·3
11	·116	·295	·0106	·0682	·00230	·00252	434	396	·123	61·0
12	·104	·264	·00849	·0548	·00287	·00314	348	318	·0989	49·0
13	·092	·234	·00665	·0429	·00367	·00402	273	250	·0774	38·4
14	·080	·203	·00503	·0324	·00485	·00530	206	188	·0585	29·0
15	·072	·183	·00407	·0263	·00599	·00657	167	153	·0474	23·5
16	·064	·163	·00322	·0208	·00752	·00839	132	120	·0374	18·6
17	·056	·142	·00246	·0159	·0099	·0108	101	91·5	·0287	14·2
18	·048	·122	·00181	·0117	·0135	·0147	74·2	67·8	·0211	10·4
19	·040	·102	·00126	·00811	·0194	·0212	51·6	47·1	·0146	7·26
20	·036	·0914	·00102	·00657	·0239	·0262	41·8	38·2	·0118	5·88
21	·032	·0813	·000804	·00519	·0304	·0331	32·9	30·1	·00936	4·64
22	·028	·0711	·000616	·00397	·0396	·0433	25·3	23·0	·00717	3·56
23	·024	·0610	·000452	·00292	·0539	·0589	18·5	17·0	·00526	2·61
24	·022	·0559	·000380	·00245	·0642	·0701	15·6	14·3	·00443	2·19
25	·020	·0508	·000314	·00203	·0778	·0849	12·8	11·8	·00366	1·80

TABLE I. (*continued.*)

Cross-section of Round Wires, with Resistance, Conductivity, and Weight of Hard-Drawn Pure Copper Wires, according to the New Standard Wire Gauge Legalised by Order in Council, August 23, 1883.

Temperature 15° Cent.

Descriptive No.	Diameter.		Area of Cross-section.		Resistance.		Conductivity.		Weight. (Density=8·95)	
	Ins.	Cms.	Sq. ins.	Sq. cms.	Legal Ohms per Yard.	Legal Ohms per Metre.	Yards per Legal Ohm.	Metres per Legal Ohm.	Lbs. per Yard.	Grms. per Metre.
26	·018	·0457	·000254	·00164	·0958	·105	10·4	9·54	·00296	1·47
27	·0164	·0417	·000211	·00136	·116	·123	8·65	7·93	·00246	1·22
28	·0148	·0376	·000172	·00111	·141	·155	7·07	6·45	·00200	·893
29	·0136	·0345	·000145	·000937	·168	·183	5·95	5·45	·00169	·839
30	·0124	·0315	·000121	·000779	·202	·221	4·86	4·53	·00141	·697
31	·0116	·0295	·000106	·000682	·230	·252	4·34	3·96	·00123	·610
32	·0108	·0274	·0000916	·000591	·266	·291	3·75	3·44	·00107	·529
33	·0100	·0254	·0000785	·000507	·311	·339	3·22	2·94	·000914	·453
34	·0092	·0234	·0000665	·000429	·367	·402	2·73	2·50	·000774	·384
35	·0084	·0213	·0000554	·000358	·440	·481	2·27	2·08	·000645	·320
36	·0076	·0193	·0000454	·000293	·540	·587	1·86	1·70	·000548	·262
37	·0068	·0173	·0000363	·000234	·672	·736	1·49	1·36	·000423	·210
38	·0060	·0152	·0000283	·000182	·862	·944	1·16	1·04	·000329	·163
39	·0052	·0132	·0000212	·000137	1·15	1·26	·870	·796	·000247	·123
40	·0048	·0122	·0000181	·000117	1·32	1·47	·759	·679	·000211	·104
41	·0044	·0112	·0000152	·0000981	1·60	1·75	·624	·570	·000177	·0878
42	·0040	·0102	·0000126	·0000811	1·94	2·13	·516	·471	·000146	·0726
43	·0036	·00914	·0000102	·0000657	2·39	2·62	·418	·382	·000118	·0588
44	·0032	·00813	·00000804	·0000519	3·04	3·32	·330	·301	·0000936	·0464
45	·0028	·00711	·00000616	·0000397	3·96	4·33	·253	·230	·0000717	·0356
46	·0024	·00610	·00000452	·0000292	5·39	5·90	·185	·170	·0000527	·0261
47	·0020	·00508	·00000314	·0000203	7·76	8·49	·128	·118	·0000366	·0181
48	·0016	·00406	·00000201	·0000130	12·20	13·3	·0824	·0754	·0000234	·0116
49	·0012	·00305	·00000113	·00000730	21·6	23·5	·0464	·0425	·0000132	·00653
50	·0010	·00254	·000000785	·00000507	31·1	33·9	·0322	·0294	·00000914	·00453

Note.—The resistances and conductivities in Tables I. and II. are calculated by taking $1{\cdot}624 \times 10^{-6}$ Legal Ohm as the resistance at 0° C., between the ends of a hard-drawn copper wire 1 cm. long and 1 sq. cm. in cross-section. This agrees with Matthiessen and Hockin's result (*B. A. Rep.*, 1864, and *Phil. Mag.*, vol. xxix., 1865) of ·1469 B.A. unit as the resistance at 0° C. of a wire one metre long weighing one gramme, if the density, 8·95, of cast specimens of their copper be taken as approximately the density of the wires experimented on, which was not determined. To reduce the numbers to accord with the B.A. unit add 1·12 per cent. to the resistances and subtract 1·12 per cent. from the conductivities.

TABLE II.

CROSS-SECTION OF ROUND WIRES, WITH RESISTANCE, CONDUCTIVITY, AND WEIGHT OF PURE COPPER WIRES, ACCORDING TO THE BIRMINGHAM WIRE GAUGE. (See *Note* to Table I.)

Temperature 15° Cent.

B. W. G.	Diameter.		Area of cross-section.		Resistance.		Conductivity.		Weight, (density = 8·95).	
	Ins.	Cms.	Sq. ins.	Sq. cms.	Legal Ohms per Yard.	Legal Ohms per Metre.	Yards per Legal Ohm.	Metres per Legal Ohm.	lbs. per Yard.	Grms. per Metre.
0000	·454	1·153	·162	1·0444	·000150	·000165	6640	6072	1·884	934·7
000	·425	1·079	·142	·915	·000172	·000188	5819	5321	1·651	819·1
00	·380	·965	·113	·732	·000215	·000235	4653	4254	1·320	654·8
0	·340	·864	·0908	·586	·000269	·000294	3735	3415	1·056	524·2
1	·300	·762	·0707	·456	·000345	·000378	2899	2652	·822	408·1
2	·284	·721	·0633	·409	·000385	·000420	2599	2377	·737	365·8
3	·259	·658	·0527	·340	·000463	·000506	2162	1996	·613	304·2
4	·238	·605	·0445	·287	·000548	·000599	1825	1669	·518	256·9
5	·220	·559	·0380	·245	·000642	·000701	1561	1427	·442	219·5
6	·203	·516	·0324	·209	·000754	·000824	1328	1214	·377	186·9
7	·180	·457	·0254	·164	·000958	·00105	1044	1004	·296	146·9
8	·165	·419	·0214	·138	·00114	·00125	877	802	·249	123·5
9	·148	·376	·0172	·111	·00141	·00155	706	645	·200	99·3
10	·134	·340	·0141	·0910	·00173	·00189	578	529	·164	81·4
11	·120	·305	·0113	·0730	·00216	·00235	463	424	·132	65·5
12	·109	·277	·00933	·0602	·00261	·00286	382	350	·109	53·9
13	·095	·241	·00709	·0457	·00344	·00376	291	266	·0825	40·9
14	·083	·211	·00541	·0349	·00451	·00492	221	203	·0630	31·2
15	·072	·183	·00407	·0263	·00599	·00655	167	153	·0474	23·5
16	·065	·165	·00331	·0214	·00735	·00804	136	124	·0386	19·2
17	·058	·147	·00264	0170	·00923	·0101	108	98·7	·0307	15·3
18	·049	·124	·00189	·0122	·0130	·0141	77·3	70·7	·0220	10·9
19	·042	·107	·00139	·00894	·0176	·0194	56·8	52·0	·0161	8·00
20	·035	·0889	·000962	·00621	·0253	·0277	39·4	36·1	·0122	5·56
21	·032	·0813	·000804	·00519	·0304	·0331	32·0	30·1	·00936	4·64
22	·028	·0711	·000616	·00397	·0395	·0433	25·3	23·1	·00716	3·55
23	·025	·0635	·000491	·00317	·0496	·0543	20·2	18·4	·00571	2·83
24	·022	·0559	·000380	·00245	·0642	·0701	15·6	14·3	·00442	2·19
25	·020	·0508	·000314	·00203	·0778	·0849	12·8	11·7	·00367	1·82
26	·018	·0457	·000254	·00164	·0959	·105	10·2	9·53	·00296	1·47
27	·016	·0406	·000201	·00130	·122	·133	8·25	7·54	·00234	1·16
28	·014	·0356	·000154	·000993	·158	·173	6·31	5·77	·00179	·889
29	·013	·0330	·000133	·000856	·184	·201	5·41	4·98	·00154	·766
30	·012	·0305	·000113	·000732	·216	·235	4·64	4·24	·00132	·653
31	·010	·0254	·0000785	·000507	·311	·339	3·23	2·95	·000915	·454
32	·009	·0229	·0000636	·000410	·384	·419	2·51	2·39	·000746	·367
33	·008	·0203	·0000503	·000324	·486	·530	2·06	1·88	·000585	·290
34	·007	·0178	·0000385	·000248	·634	·693	1·58	1·45	·000442	·220
35	·005	·0127	·0000196	·000127	1·25	1·35	·806	·736	·000229	·113
36	·004	·0102	·0000126	·0000811	1·94	2·13	·516	·471	·000146	·0726

TABLE III.

Conductivities of Pure Metals at $t°$ C.*

Conductivity at 0° = 1.

Metal.	Conductivity at $t°$ C.
Silver	$1 - \cdot0038278t + \cdot000009848t^2$
Copper	$1 - \cdot0038701t + \cdot000009009t^2$
Gold	$1 - \cdot0036745t + \cdot000008443t^2$
Zinc	$1 - \cdot0037047t + \cdot000008274t^2$
Cadmium	$1 - \cdot0036871t + \cdot000007575t^2$
Tin	$1 - \cdot0036029t + \cdot000006136t^2$
Lead	$1 - \cdot0038756t + \cdot000009146t^2$
Arsenic	$1 - \cdot0038996t + \cdot000008879t^2$
Antimony	$1 - \cdot0030826t + \cdot000010364t^2$
Bismuth	$1 - \cdot0035216t + \cdot000005728t^2$
Iron	$1 - \cdot0051182t + \cdot000012916t^2$

* From the results of Matthiessen's experiments; and to be used only for temperatures between 0°C. and 100° C. The formulas, excluding that for iron, agree closely, and give the mean formula $1 - \cdot0037647, + \cdot00008340t^2$.

TABLE IV.

Conductivity and Resistance of Pure Copper at Temperatures from 0° C. to 40°C.

Calculated by the formula for the Conductivity of Copper in Table III.

Temp.	Conductivity.	Resistance.	Temp.	Conductivity.	Resistance.
0°	1·0000	1·0000	21°	0·9227	1·0838
1	0·9961	1·00388	22	0·9192	1·0879
2	0·9923	1·00776	23	0 9158	1·0920
3	0·9885	1·0116	24	0·9123	1·0961
4	0·9847	1·0156	25	0·9089	1·1003
5	0·9809	1·0195	26	0·9054	1·1044
6	0·9771	1·0234	27	0·9020	1·8085
7	0·9734	1·0274	28	0·8987	1·1127
8	0·9696	1·0313	29	0·8953	1·1169
9	0·9559	1·0353	30	0·8920	1·1211
10	0·9622	1·0393	31	0·8887	1·1253
11	0·9585	1·0433	32	0·8854	1·1295
12	0·9549	1·0473	33	0·8821	1·1337
13	0·9512	1·0513	34	0·8788	1·1379
14	0·9476	1·0553	35	0·8756	1·1421
15	0·9440	1·0593	36	0·8723	1·1464
16	0·9404	1·0634	37	0·8691	1·1506
17	0·9368	1·0675	38	0·8659	1·1548
18	0·9333	1·0715	39	0·8628	1·1591
19	0·9297	1·0756	40	0·8596	1·1633
20	0·9262	1·0797			

Specific Resistances in Legal Ohms of Wires of Different Metals and Alloys.[1]

Substance.	Resistance at 0° C. of wire one cm. long, one sq. cm. in section.	Resistance at 0° C. of a wire one foot long weighing one grain.	Resistance at 0° C. of a wire one metre long weighing one gramme.	Resistance at 0° C. of a wire one foot long, 1 – 1000th in. in diam.	Resistance at 0° C. of a wire one metre long one millimetre in diam.	Percentage increase of resistance for 1° C. increase of temperature at 20° C.
Silver, annealed	1.504×10^{-6}	0·01916	0·1527	9·049	·2190	0·377
Silver, hard drawn	1·534 ,,	0·02080	0·1661	9·825	·2388	...
Copper, annealed	1·598 ,,	0·02034	0·1424	9·609	·2041	0·388
Copper, hard drawn	1·634 ,,	0·02081	0·1453	9·829	·2083	...
Gold, annealed	2·058 ,,	0·02621	0·4035	12·38	·5784	0·365
Gold, hard drawn	2·095 ,,	0·02667	0·4104	12·60	·5883	...
Aluminium, annealed	2·912 ,,	0·03710	0·0749	17·52	·1073	...
Zinc, pressed	5·614 ,,	0·07163	0·4022	33·84	·5766	0·365
Platinum, annealed	9·055 ,,	0·1153	1·94	54·47	2·779	...
Iron, annealed	9·715 ,,	0·1237	0·7570	58·44	1·085	...
Nickel, annealed	12·46 ,,	0·1586	1·059	74·93	1·518	...
Tin, pressed	13·21 ,,	0·1682	0·9629	79·46	1·381	0·365
Lead, pressed	19·63 ,,	0·2498	2·232	118·05	3·200	0·387
Antimony, pressed	35·50 ,,	0·4520	2·384	214	3·418	0·389
Bismuth, pressed	131·2 ,,	1·670	12·89	789	18·44	0·354
Mercury, liquid (see *Note*)	95·11 ,,	1·2112	12·92	572·1	18·51	0·072
Platinum - Silver Alloy,[2] hard or annealed	24·39 ,,	0·3105	2·926	146·69	4·195	0·031
German Silver Alloy, hard or annealed	20·99 ,,	0·2665	1·83	125·89	2·623	0·044
Gold-Silver Alloy,[3] hard or annealed	10·87 ,,	0·1384	1·650	65·36	2·365	0·065
Platinoid[1]	32·8 ,,	...	...	...	...	·021
Hadfield's manganese steel[1]	68· about	...	...	...	...	·122

[1] Reduced (with the exception of platinoid and manganese steel) from a table given by Professor Jenkin as expressing the results of Matthiessen's experiments. The numbers for platinoid and Hadfield's manganese steel are taken from a paper by Professor J. A. Fleming (*Electrician*, March 9, 1888). The percentage variation of resistance for these two substances is the average for the range between 0° C. and 100° C.

[2] Two parts platinum, one part silver, by weight.

[3] Two parts gold, one part silver, by weight.

Note.—According to a very careful determination of the specific resistance of mercury made by Lord Rayleigh and Mrs. Sidgwick (*Phil. Trans.*, Part I., 1883, and above, p. 389), the value given in this table is about ·8 per cent. too high. Their final result is 95.412×10^{-6} B.A. unit as the resistance at 0° C. of a column of pure mercury one cm. long and one sq. cm. in section. A column of pure mercury therefore one sq. millimetre in section, which at 0° C. has a resistance of one ohm, has, according to the B.A. determination of the ohm, a length of 104·81 cms., and according to Lord Rayleigh and Mrs. Sidgwick's determination, 106·21 cms. According to the legal ohm the length is, as stated above, p. 315, 106 cms.

The value given in Col. I. for hard-drawn copper has evidently been calculated from the corresponding observed result in Col. III., by using the density 8·89 for copper, and is therefore higher than that on which Tables I. and II. are founded. (See *Note* to Table I.)

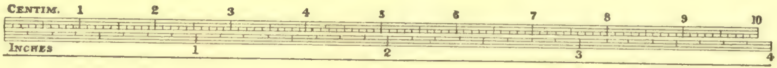
Metric System of Weights and Measures. (From Messrs. De la Rue and Co.'s Diaries).

FRENCH MEASURES OF LENGTH.

	In English Inches.	In English Feet = 12 Inches.	In English Yards = 3 Feet.	In English Fathoms = 6 Feet.	In English Miles = 1,760 Yards.
Millimètre	0·03937	0·003281	0·0010936	0·0005468	0·0000006
Centimètre	0·39371	0·032809	0·0109363	0·0054682	0·0000062
Décimètre	3·93708	0·328090	0·1093633	0·0546816	0·0000621
Mètre	39·37079	3·280899	1·0936331	0·5468165	0·0006214
Décamètre	393·70790	32·808992	10·9363306	5·4681653	0·0062138
Hectomètre	3937·07900	328·089917	109·3633056	54·6816528	0·0621382
Kilomètre	39370·79000	3280·899167	1093·6330556	546·8165278	0·6213834
Myriamètre	393707·90000	32808·991667	10936·3305556	5468·1952778	6·2138242

1 Inch = 2·539954 Centimètres.
1 Foot = 3·047944 Décimètres.

1 Yard = 0·9143835 Mètre.
1 Mile = 1·6093149 Kilomètre.



FRENCH MEASURES OF SURFACE.

	In English Square Feet.	In English Sq. Yards = 9 Square Feet.	In English Poles = 272·25 Sq. Feet.	In English Roods = 10890 Sq. Feet.	In English Acres = 43560 Sq. Feet.
Centiare or square mètre	10·764299	1·196033	0·0395383	0·0009885	0·0002471
Are, or 100 square mètres	1076·429934	119·603326	3·9538290	0·0988457	0·0247111
Hectare, or 10,000 square mètres	107642·993419	11960·332602	395·3828959	9·8845724	2·4711431

1 Square Inch = 6·4513669 Square Centimètres.
1 Square Foot = 9·2899683 Square Décimètres.

1 Square Yard = 0·83609715 Square Mètre or Centiare.
1 Acre = 0·40467102 Hectare.

1 Square Mile = 2·58989451 Square Kilomètres.

METRIC SYSTEM OF WEIGHTS AND MEASURES.

FRENCH MEASURES OF CAPACITY.

	In Cubic Inches	In Cubic Feet = 1728 Cubic Inches.	In Pints = 34·65923 Cubic Inches.	In Gallons = 8 Pints. = 277·27384 Cubic Inches.	In Bushels = 8 Gallons = 2218·19075 Cubic Inches.
Millilitre, or cubic centimètre	0·06103	0 000035	0·00176	0·0002201	0·0000275
Centilitre, or 10 cubic centimètres..	0·61027	0·000353	0·01761	0·0022010	0·0002751
Décilitre, or 100 cubic centimètres..	6·10271	0·003532	0·17608	0·0220097	0·0027512
Litre, or cubic décimètre	61·02705	0·035317	1·76077	0·2200967	0·0275121
Décalitre, or centistère	610·27052	0·353166	17·60773	2·2009668	0·2751208
Hectolitre, or décistère	6102·70515	3·531658	176·07734	22·0096677	2·75120185
Kilolitre, or Stère, or cubic mètre....	61027·05152	35·316581	1760·77344	220·0966767	27·5120846
Myrialitre, or décastère............	610270·51519	353·165807	17607·73437	2200·9667675	275·1208459

1 Cubic Inch = 16·386176 Cubic Centimètres. 1 Cubic Foot = 28·315312 Cubic Décimètres.
1 Gallon = 4·543458 Litres.

FRENCH MEASURES OF WEIGHT.

	In English Grains.	In Troy Ounces = 480 Grains.	In Avoirdupois Lbs. = 7,000 Grains.	In Cwts. = 112 Lbs. = 784000 Grains.	In Tons = 20 Cwt. = 15680000 Grains.
Milligramme	0·01543	0·000032	0·0000022	0·0000000	0·0000000
Centigramme............	0·15432	0 000322	0·0000220	0·0000002	0·0000000
Décigramme	1·54323	0·003215	0·0002205	0·0000020	0·0000001
Gramme	15·43235	0·032151	0·0022046	0·0000197	0·0000010
Décagramme	154·32349	0·321507	0·0220462	0·0001968	0·0000098
Hectogramme	1543·23488	3·215073	0·2204621	0·0019684	0·0000984
Kilogramme	15432·34880	32·150727	2·2046213	0·0196841	0·0009842
Myriagramme	154323·48800	321·507267	22·0462126	0·1968412	0·0098421

1 Grain = 0·064799 Gram. | 1 Lb. Avd. = 0·453593 Kilogr.
1 Troy oz. = 31·103496 Grams. | 1 Cwt. = 50·802377 Kilogrs.

TABLE VII.

For reduction of Period of Oscillation observed for finite amplitude to Period for infinitely small amplitude (see p. 223 above). If T be the observed period and $1 - k$ the reducing factor, so that kT is to be subtracted, the values of k are as follows :—

Amplitude.	k	Amplitude.	k
0	·00000	11	·00230
1	·00002	12	·00274
2	·00008	13	·00322
3	·00017	14	·00373
4	·00030	15	·00428
5	·00048	16	·00487
6	·00069	17	·00550
7	·00093	18	·00616
8	·00122	19	·00686
9	·00154	20	·00761
10	·00190		

TABLE VIII.

Units of Work or Energy.

1 erg	$2{\cdot}374 \times 10^{-6}$ foot-poundal.
„	$7{\cdot}375 \times 10^{-8}$ foot-pound at London.
1 centimetre-gramme at Paris	981 ergs.
„ „ „ at London	981·17 ergs
„ „ „ „ „	$2{\cdot}329 \times 10^{-3}$ foot-poundal.
1 metre-kilogramme at Paris	981×10^5 ergs.
„ „ „ at London	$981{\cdot}17 \times 10^5$ ergs.
„ „ „ „ „	7·236 foot-pound.
1 foot-poundal	421390 ergs.
1 foot-pound at London . . .	$13{\cdot}56 \times 10^6$ ergs.
1 *Joule*	10^7 ergs.

TABLE IX.

Units of Activity or Rate of Working.

1 erg per second	$1{\cdot}34 \times 10^{-10}$ horse-power at London.
1 horse-power	33000 foot-pounds per minute.
„ „ at London . . .	$7{\cdot}46 \times 10^{9}$ ergs per second.
1 force-de-cheval at Paris . .	$7{\cdot}36 \times 10^{9}$ ergs per second.
1 *Watt*	10^{7} ergs per second.

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END OF VOL. I.

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